5. Analytical Analysis of the Integrated Model

The system of the integrated model consists of two state and two decision variables (see table 4.2) and is thus much more complex than the one discussed in chapter 2. This complicates an analytical analysis significantly and we can only prove structural results under restrictive assumptions. We start off by analyzing the single-period problem (newsvendor model) and then extend some of the obtained results to the two-period case. However, even in this simplest version of the model, we only consider loss neutral customer behavior \((\beta_2 = \beta_3)\) to ensure analytical tractability. Thus the demand model (4.2.1) reduces to

\[
D(p_t, r_t, \epsilon_t) = \beta_0 + \beta_1 \cdot p_t + \beta_2(p_t - r_t) + \epsilon_t. \tag{5.0.1}
\]

Furthermore, for section 5.2 we assume that the random variable \(\epsilon\), which follows an arbitrary probability function \(f(\cdot)\), is continuous and differentiable in order to avoid additional complexity of the analytical analysis for the two period model. At the end of this chapter, for the multi-period case we give an extension of the proof of Federgruen and Heching (1999) and show the optimality of a base-stock policy in the integrated model. After all, those results are not of extreme practical relevance, since several assumptions on the demand and revenues have to be made, which many commonly used demand functions including the linear one, defined in equation (5.0.1), do not fulfill. However, we will provide an extensive numerical study in chapter 6, where we show that in the cases analyzed the obtained results still hold under much less restrictive assumptions.

5.1. One-period model

We will start our analysis of equation (4.3.1) with the last period. The optimality of a base-stock-list-price policy follows directly from Federgruen and Heching (1999), since the reference price \(r\) is only an additional parameter for the one period case. In this section however, we will provide an alternative proof and give implicit solutions for the optimal price and inventory level with respect to reference price, which will be used to extend the base-stock-list-price result to a two-period setting in section 5.2. To simplify the notation we in the following write, where not stated differently, \(D\) for \(D(p, r, \epsilon)\) and \(E[D]\) for \(E[D(p, r, \epsilon)]\). Furthermore \(E[D_y], E[D_p], E[D_r]\) denote the derivatives of expected demand \(E[D]\) with
respect to $y$, $p$, $r$. According to chapter 4, the expected one-period profit is given by

$$E[\Pi(x, y, p, r, \epsilon)] = pE[D] - c(y - x) - G(y, p, r),$$

(5.1.1)

with expected holding and backlogging costs

$$G(y, p, r) = h \int_{-\infty}^{y - E[D]} (y - E[D] - u)f(u)du + b \int_{y - E[D]}^{\infty} (E[D] + u - y)f(u)du. \quad (5.1.2)$$

**Lemma 5.1.** The expected profit $E[\Pi(x, y, p, r, \epsilon)]$ is jointly concave and submodular (see definition 2.3) in $y$ and $p$. Furthermore, $E[\Pi(x, y, p, r, \epsilon)]$ is strictly concave in $p$.

**Proof.** By applying Leibniz’s integration rule we obtain the following partial derivatives:

$$\frac{\partial E[\Pi(x, y, p, r, \epsilon)]}{\partial y} = (b - c) - (h + b)F(y - E[D]),$$

(5.1.3)

$$\frac{\partial^2 E[\Pi(x, y, p, r, \epsilon)]}{\partial y^2} = -(h + b)f(y - E[D]),$$

(5.1.4)

$$\frac{\partial E[\Pi(x, y, p, r, \epsilon)]}{\partial p} = E[D] + (p - b)E[D_p] + (h + b)E[D_p]F(y - E[D]),$$

(5.1.5)

$$\frac{\partial^2 E[\Pi(x, y, p, r, \epsilon)]}{\partial p^2} = 2E[D_p] + (p - b)E[D_{pp}] - (h + b)E[D_p]^2 f(y - E[D]) +$$

$$+ (h + b)E[D_{pp}]F(y - E[D]),$$

(5.1.6)

$$\frac{\partial^2 E[\Pi(x, y, p, r, \epsilon)]}{\partial p \partial y} = (h + b)E[D_p]f(y - E[D]).$$

(5.1.7)

From equation (4.2.1), it is easy to see that $\frac{\partial^2 E[\Pi(x, y, p, r, \epsilon)]}{\partial y^2} \leq 0$ and $\frac{\partial^2 E[\Pi(x, y, p, r, \epsilon)]}{\partial p^2} < 0$ and hence the expected profit $E[\Pi(x, y, p, r, \epsilon)]$ is concave in $y$ and strictly concave in $p$. Furthermore, $E[\Pi(x, y, p, r, \epsilon)]$ is submodular in $y$ and $p$ by definition 2.3, since $\frac{\partial^2 E[\Pi(x, y, p, r, \epsilon)]}{\partial p \partial y} \leq 0$. Moreover the determinant of the Hesse matrix is

$$\frac{\partial^2 E[\Pi(x, y, p, r, \epsilon)]}{\partial y^2} \frac{\partial^2 E[\Pi(x, y, p, r, \epsilon)]}{\partial p^2} - \frac{\partial^2 E[\Pi(x, y, p, r, \epsilon)]}{\partial p \partial y} =$$

$$= -(h + b)f(y - E[D]) \cdot (2E[D_p] + (p - b)E[D_{pp}] + (h + b)E[D_{pp}]F(y - E[D])) > 0.$$  

(5.1.8)

Hence, the Hesse matrix is positive definite, which ensures that $E[\Pi(x, y, p, r, \epsilon)]$ is jointly concave in $p$ and $y$. \hfill \Box

For later investigations we also need some structural properties of the expected holding and backlogging cost function $G(y, p, r)$, which we provide in the following.
Lemma 5.2. The expected holding and backlogging costs \( G(y, p, r) \) defined in equation (5.1.2) are convex in \( y \) but not necessarily in \( p \). Furthermore, \( G(y, p, r) \) is supermodular in \( y \) and \( p \).

**Proof.** The partial derivatives of \( G(y, p, r) \) with respect to \( y \) and \( p \) are given by:

\[
\frac{\partial G(y, p, r)}{\partial y} = -b + (h + b)F(y - E[D]), \tag{5.1.9}
\]

\[
\frac{\partial^2 G(y, p, r)}{\partial y^2} = (h + b)f(y - E[D]) \geq 0, \tag{5.1.10}
\]

\[
\frac{\partial G(y, p, r)}{\partial p} = bE[D_p] - (h + b)E[D_p]F(y - E[D]). \tag{5.1.11}
\]

\[
\frac{\partial^2 G(y, p, r)}{\partial p \partial y} = bE[D_{pp}] + (h + b)E[D_p]f(y - E[D]) - (h + b)E[D_{pp}]F(y - E[D]). \tag{5.1.12}
\]

Equation (4.2.1) ensures that \( G(y, p, r) \) is convex in \( y \). Note that \( G(y, p, r) \) is not necessarily convex in \( p \). Furthermore,

\[
\frac{\partial^2 G(y, p, r)}{\partial p \partial y} = -(h + b)E[D_p]f(y - E[D]) \geq 0, \tag{5.1.13}
\]

which ensures the supermodularity in \( p \) and \( y \).

The above two lemmas lead to an optimal pricing and ordering policy (compare section 2.2).

**Theorem 5.1** (Base-stock-list-price policy). For the linear demand function defined in equation (5.0.1) and the system (4.3.1) to (4.3.3) and \( b > (1 - \gamma)c \), the optimal policy for the one-period case is a base-stock-list-price policy, where \( y^*(x, r) \) and \( p^*(x, r) \) are given by

\[
y^*(x, r) = \begin{cases} 
S^*(P^*(r), r) & , x < S^*(P^*(r), r) \\
x & , \text{else}
\end{cases} \tag{5.1.14}
\]

\[
p^*(x, r) = \begin{cases} 
P^*(r) & , x < S^*(P^*(r), r) \\
p^*(x, r) & , \text{else}
\end{cases} \tag{5.1.15}
\]

where the base-stock level \( S^*(P^*(r), r) \) is given by

\[
S^*(P^*(r), r) = F^{-1}\left(\frac{b - (1 - \gamma)c}{h + b}\right) + E[D(P^*(r), r, \epsilon)]. \tag{5.1.16}
\]

and the list-price \( P^*(r) \) is the unique solution to

\[
E[D(P^*(r), r, \epsilon)] + (P^*(r) - c)E[D_p(P^*(r), r, \epsilon)] = 0 \tag{5.1.17}
\]
Furthermore, the discounted price $p^*(x, r)$ is given implicitly by

$$E[D(p^*(x, r), r, \epsilon)] + (p^*(x, r) - \gamma c - b)E[D_p(p^*(x, r), r, \epsilon)] + (h + b)E[D_p(p^*(x, r), r, \epsilon)]F(x - E[D(p^*(x, r), r, \epsilon)]) = 0. \quad (5.1.18)$$

**Proof.** The expected total profit $J(x, y, p, r)$ from equation (4.3.2) can be expressed in terms of the expected one-period profit $E[\Pi(x, y, p, r, \epsilon)]$ such that

$$J(x, y, p, r) = E[\Pi(x, y, p, r, \epsilon)] + \gamma c(y - E[D(p, r, \epsilon)]). \quad (5.1.19)$$

Since $y - E[D(p, r, \epsilon)]$ is trivially jointly concave in $y$ and $p$, it follows directly from lemma 5.1 that the expected total profit $J(x, y, p, r)$ is jointly concave in $y$ and $p$. Thus the optimization problem over the two variables $y$ and $p$ can be reduced to an optimization problem over the single variable $y$ as a function of $p$ with subsequent substitution of the result back into $J(x, y, p, r)$, which is thereafter solved for $p$. We in the following show the optimality of a base-stock policy and provide an optimal order-up-to level $S^*(r)$ as a function of price $p$ and then continue with the price optimization. By setting equation

$$\frac{\partial J(x, y, p, r)}{\partial y} = (b - (1 - \gamma)c) - (h + b)F(y - E[D])] \quad (5.1.20)$$

equal to zero we obtain the solution to $\max_y J(x, y, p, r)$ which is denoted by

$$y(x, r, p) = F^{-1}\left(\frac{b - (1 - \gamma)c}{h + b}\right) + E[D(p, r, \epsilon)]. \quad (5.1.21)$$

Since $y(x, r, p)$ is not necessarily greater or equal to $x$, but the model only allows for non-negative orders ($y \geq x$), (5.1.21) gives the optimal solution to $\max_{y \geq x} J(x, y, p, r)$ only in the case $y(x, r, p) \geq x$. In the case of $y(x, r, p) < x$ the optimal policy is not to order (argmax$_{y \geq x} J(x, y, p, r) = x$), since $J(x, y, p, r)$ is concave in $y$. Thus a base-stock policy (compare definition 2.1) with order-up-to-level

$$S^*(p, r) = F^{-1}\left(\frac{b - (1 - \gamma)c}{h + b}\right) + E[D(p, r, \epsilon)] \quad (5.1.22)$$
is optimal.

For proving equations (5.1.17) and (5.1.18) we now need to distinguish between the two cases $x < S^*(p, r)$ and $x \geq S^*(p, r)$.

Let $x < S^*(p, r)$. For notational convenience we denote $y^0 := F^{-1}\left(\frac{b - (1 - \gamma)c}{h + b}\right)$, which yields $S^*(p, r) = y^0 + E[D(p, r, \epsilon)]$. In order to find the optimal list-price $p$, we substitute
5.1. ONE-PERIOD MODEL

\[ y = S^*(p, r) \] into \( J(x, y, p, r) \) which by using equation (5.1.1) and equation (5.1.19) gives

\[
J(x, S^*(p, r), p, r) = pE[D(p, r, \epsilon)] - c((1 - \gamma)y^0 + E[D(p, r, \epsilon)] - x) - \\
- h \int_{-\infty}^{y^*} (y^0 - u)f(u)du - b \int_{y^*}^{\infty} (u - y^0)f(u)du.
\] (5.1.23)

By differentiating (5.1.23) with respect to \( p \) we obtain

\[
\frac{\partial J(x, S^*(p, r), p, r)}{\partial p} = E[D(p, r, \epsilon)] + (p - c)E[D_p(p, r, \epsilon)].
\] (5.1.24)

Since \( J(x, y, p, r) \) is jointly concave in \( p \) and \( y \), equation (5.1.17) follows directly by setting equation (5.1.24) equal to zero.

Let \( x > S^*(p, r) \) which yields \( y^*(x, r, p) = x \). Then substituting \( y = x \) gives

\[
J(x, x, p, r) = (p - \gamma c)E[D(p, r, \epsilon)] + \gamma cx - \\
- h \int_{-\infty}^{x - E[D(p, r, \epsilon)]} (x - E[D(p, r, \epsilon)] - u)f(u)du - \\
- b \int_{x - E[D(p, r, \epsilon)]}^{\infty} (E[D(p, r, \epsilon)] + u - x)f(u)du.
\] (5.1.25)

Differentiation with respect to \( p \) results in

\[
\frac{\partial J(x, x, p, r)}{\partial p} = E[D(p, r, \epsilon)] + (p - \gamma c - b)E[D_p(p, r, \epsilon)] + \\
+ (h + b)E[D_p(p, r, \epsilon)]F(x - E[D(p, r, \epsilon)]).
\] (5.1.26)

Since \( J(x, y, p, r) \) is jointly concave in \( p \) and \( y \), equation (5.1.18) results from setting equation (5.1.26) equal to zero.

In order to prove the optimality of a list-price policy, we need to show that \( p^*(x, r) \) is unique and non-increasing in \( x \). By equation (5.1.19) and demand \( D(p, r, \epsilon) \) being concave in \( p \) it follows by lemma 5.1 that \( J(x, y, p, r) \) is strictly concave in \( p \) and thus the optimal price \( p^*(x, r) \) is unique. Furthermore, \( J(x, y, p, r) \) is submodular in \( y \) and \( p \). It follows from theorem 2.8.1 in Topkis (1998) that the optimal price \( p^*(y, r) \) is non-increasing in \( y \) and hence in \( x \). Substituting the optimal list price \( p = P^*(r) \) in equation (5.1.22) verifies equation (5.1.16).

Remark 5.1. By the assumption \( b > (1 - \gamma)c \) in theorem 5.1 we exclude the trivial solution of not placing any orders and accumulating backlogging costs until the end of the planning horizon \( T \) (compare section 2.2.1). Thus \( F^{-1} \left( \frac{b - (1 - \gamma)c}{h + b} \right) \) is well defined and bounded by \([0, 1]\).

Remark 5.2. Note that lemma 5.1 and 5.2 as well as theorem 5.1 also hold for more general demand functions \( D(p, r, \epsilon) \), which are decreasing in price \( p \), non-decreasing in reference
price \( r \) and concave in both \( p \) and \( r \). The demand function defined in (4.2.1) of course fulfills these assumptions.

Figure 5.1 and figure 5.2 give a graphical description of the policy, which we showed to be optimal in the above theorem. They are an extension to figure 2.1 and figure 2.7 in chapter 2, with the difference that now the optimal decisions depend on two states: the inventory before ordering \( x \) and the reference price \( r \). Note that the dependency of the optimal price \( p^* \) and optimal inventory level after ordering \( y^* \) on reference price \( r \) adds an additional dimension to the solution space. It becomes clear that in contrast to chapter 2, the base-stock level \( S^*(r) \) and the optimal price \( p^*(x, r) \) depend on the consumers' price expectation \( r \).

Looking at figures 5.1 and 5.2, the question arises whether new structural properties of the optimal policies in the reference price \( r \) can be formulated. This leads to the following theorem, where for the one period case and loss-neutral customer behavior we show that both the optimal pricing and ordering policy are non-decreasing in the reference price \( r \).

To prove theorem 5.2 we introduce the theory of implicit differentiation (Heuser 1981, theorem 170.1) in the lemma below.

**Lemma 5.3 (Implicit differentiation).** Let \( G \subset \mathbb{R}^n \) and \( H \subset \mathbb{R}^m \) nonempty open sets and \( \xi \in G \) and \( \eta \in H \). Furthermore let \( F : G \times H \rightarrow \mathbb{R}^m \) be a continuous function with \( F'(\xi, \eta) = 0 \), \( F'\xi, \eta) \) is well defined and \( \frac{\partial F}{\partial y}(\xi, \eta) \) invertible. If there exists a \( \delta \)-neighborhood \( U \subset G \) of \( \xi \), an \( \epsilon \)-neighborhood \( V \subset H \) of \( \eta \) and a continuous function \( f : U \rightarrow V \) with \( f(\xi) = \eta \) and \( F(x, f(x)) = 0 \) for all \( x \in U \) then \( f \) is differentiable at \( \xi \) and \( f'\xi \) is given by

\[
f'(\xi) = - \left( \frac{\partial F}{\partial y}(\xi, \eta) \right)^{-1} \frac{\partial F}{\partial x}(\xi, \eta).
\]
5.1. ONE-PERIOD MODEL

Theorem 5.2. For the linear demand function (5.0.1) and the system defined in (4.3.1) to (4.3.3), both the optimal base-stock level \( S^*(p^*, r) \) and the optimal pricing policy \( p^*(x, r) \) are non-decreasing in the reference price \( r \).

Proof. Since for the linear demand function (4.2.1) the second and mixed partial derivatives of demand \( D(p, r, \epsilon) \) are zero (\( D_{pp}(p, r, \epsilon) = 0, D_{rr}(p, r, \epsilon) = 0, D_{pr}(p, r, \epsilon) = 0 \)), we by lemma 5.3 obtain the partial derivative of the optimal price \( p^*(x, r) \):

\[
\frac{\partial p^*(x, r)}{\partial r} = \begin{cases} 
-\frac{E[D_x]}{2E[D_p]} & x < S^*(P^*(r), r) \\
\frac{E[D_r]-E[D_p]E[D_x]}{E[D_p]-(h+b)E[D_x]} & x > S^*(P^*(r), r) 
\end{cases}
\]

(5.1.28)

where we use the short notation \( D = D(p^*(x, r), r, \epsilon) \), \( D_p = D_p(p^*(x, r), r, \epsilon) \) and \( D_r = D_r(p^*(x, r), r, \epsilon) \). Since \( D_p(p, r, \epsilon) < 0 \) and \( D_r(p, r, \epsilon) \geq 0 \) for all \( p \) and \( r \), it is easy to see that equation (5.1.28) is always greater than zero and thus the optimal price \( p^*(x, r) \) is non-decreasing in the reference price \( r \). From equations (5.1.16) and (5.1.28) it follows that the base-stock \( S^*(p^*, r) \) is also non-decreasing in \( r \):

\[
\frac{\partial S^*(p^*, r)}{\partial r} = -E[D_p(P^*(r), r, \epsilon)] \frac{E[D_r(P^*(r), r, \epsilon)]}{2E[D_p(P^*(r), r, \epsilon)]} + E[D_r(P^*(r), r, \epsilon)] \\
\geq 0.
\]

(5.1.29)

Remark 5.3. Theorem 5.2 shows that for loss-neutral customer behavior (\( \beta_2 = \beta_3 \)), the optimal price \( p^*(x, r) \) as well as the base-stock level \( S^*(r) \) are non-decreasing in the reference price.
As illustrated in figure 5.3, one can see that this result can be extended to loss-averse customer behavior for the optimal price $p^*(x,r)$. However, the base-stock $S^*(r)$ is only increasing in reference price in the loss neutral case. This is not true in the case of loss-aversion, where the base-stock $S^*(r)$ is not monotonous in reference price (see figure 5.4). While in the loss-neutral case the list-price $P^*(r)$ is a smooth (continuous) function in $r$, since expected demand is a smooth function in $p$ and $r$ due to $\beta_2 = \beta_3$, in the loss-averse case expected demand is a kinked function in $p$ and $r$ (with two different slopes depending on $p \leq r$ or $p > r$, respectively) and therefore the list-price $P^*(r)$ is a kinked function in $r$. Note that this relation results in a list-price $P^*(r)$ that is threefold: $P^*(r) < r$, $P^*(r) = r$ and $P^*(r) > r$. It is clear that for all reference prices $r$ with $P^*(r) = r$ the corresponding base-stock level $S^*(r, r)$ from equation (5.1.22) is decreasing, since

$$\frac{\partial S^*}{\partial r}(r, r) = \beta_1 < 0$$

(5.1.30)

and thus an optimal inventory policy is no longer monotonous in reference price $r$ for loss-averse customer behavior (see figure 5.4).

From the above remark it becomes clear that loss-averse customer behavior already adds considerable complexity to the model in the one-period case and thus significant additional dynamics in the multi-period setting. In this thesis we mainly concentrate on the loss neutral case.
5.2. TWO-PERIOD MODEL

We are now in the position to add one period to the planning horizon and study the two period case. This is not only a theoretical exercise but also has significant practical relevance on its own: due to shortening life-cycles an increasing fraction of a retailer’s assortment consists of products where there is only one reorder possibility. Due to long production lead times there is as a consequence only one possibility for reordering after the initial order is placed, which motivates the application of a two-time-period model.

The dynamic program defined in (4.3.1) to (4.3.3) will be used in this section for \( t = 2 \). Corresponding to (4.3.4) and (4.3.5) the transition functions are given by

\[
X_2 = x_2(y_1, p_1, r_1, c_1) = y_1 - D_1(p_1, r_1, c_1),
\]

\[
r_2 = r_2(p_1, r_1) = \alpha r_1 + (1 - \alpha)p_1.
\]

where the subscript '1' denotes the first time period and '2' the second time period. Thus \( V_2(x_2, r_2) \) is rewritten as \( V_2(x_2(y_1, p_1, r_1, c_1), r_2(p_1, r_1)) \).

In Federgruen and Heching (1999), joint concavity of the value function \( J_1(x_t, y_t, p_t) \) is used to show the optimality of a base-stock policy. However, this approach does not work in the case of reference prices. In this case, \( V_2(x_2(y_1, p_1, r_1, \epsilon_1), r_2(p_1, r_1)) \) is not necessarily concave in \( p_1 \) and therefore we cannot conclude with further investigations that \( J_1(x_1, y_1, p_1) \) is jointly concave in \( y_1 \) and \( p_1 \).

**Lemma 5.4.** For the linear demand function (5.0.1), the system defined in (4.3.1) to (4.3.3) and \( T = 2 \), the value function of the second time period \( v_2(x_2(y_1, p_1, r_1, \epsilon_1), r_2(p_1, r_1)) \) is not necessarily concave in the selling price of the first time period \( p_1 \).
Proof. Since $J(x_2, y_2, p_2, r_2) = E[\Pi(x_2, y_2, p_2, r_2)] + \gamma c(y - E[D(p_2, r_2, \epsilon_2)])$, the expected profit of the last time-period $J_2(x_2, y_2, p_2, r_2)$ is jointly concave in inventory level $y_2$ and price $p_2$ by lemma 5.1 and thus a base-stock-policy is optimal (see theorem 5.1). Hence, the last period's optimal profit can be rewritten by substituting $h$ by $h - \gamma c$ and $b$ by $b + \gamma c$:

$$v_2(x_2, r_2) = \begin{cases} V_2^*(r_2), & x_2 \leq S_2^*(r_2) \\ \mathcal{W}_2(x_2, r_2), & \text{else} \end{cases}$$

(5.2.3)

Substituting the transition functions (5.2.1) and (5.2.2), the optimal value function becomes $v_2(x_2, r_2) = v_2(y_1, p_1, r_1, \epsilon_1, r_2(p_1, r_1))$. Note that the holding costs in the second time-period are given by $h - \gamma c$, and the backlogging costs by $b + \gamma c$, respectively (compare page 22). Thus by substituting $h$ by $h - \gamma c$ and $b$ by $b + \gamma c$, the total optimal profit $v_2(x_2, r_2)$ is given by $E[\Pi_2(x_2, y_2, p_2, r_2), r_2(p_1, r_1), \epsilon_2)]$.

Although $V_2^*(r_2) = E[\Pi_2(x_2, S^*(r_2), P^*(r_2), r_2, \epsilon_2)]$ is concave in $p_1$ by lemma A.2 in the appendix A, this is not true for $\mathcal{W}_2(x_2, r_2) = E[\Pi_2(x_2, x_2, p_2^*(x_2, r_2), r_2, \epsilon_2)]$. Lemma A.3 shows that $\partial^2 \mathcal{W}_2(x_2, r_2)/\partial p_1^2$ is not necessarily less than zero. Hence, concavity is not guaranteed for $\mathcal{W}_2(x_2, r_2)$ in $p_1$ and thus we cannot conclude that the optimal profit of the second time period $v_2(x_2(y_1, p_1, r_1, \epsilon_1, r_2(p_1, r_1)))$ is concave in $p_1$.

In order to prove that a base-stock policy is still optimal in the first-time period, we will show that $J_1(y_1, p_1, x_1, r_1)$ is jointly concave in $y_1$ and $p_1$, although $V_2(x_2, r_2)$ is not jointly concave in $y_1$ and $p_1$ (see lemma 5.4). Using equation (4.3.2), the expected profit of the first period can be written as

$$J_1(x_1, y_1, p_1, r_1) = E[\Pi_1(x_1, y_1, p_1, r_1, \epsilon_1)] + \gamma E[V_2(x_2(y_1, p_1, r_1, \epsilon_1), r_2(p_1, r_1))].$$
5.2. TWO-PERIOD MODEL

By using the short notation \( E[D(p_1,r_1,\epsilon_1)] = E[D] \), profit of the first time period then becomes

\[
\Pi_1(x_1,y_1,p_1,r_1,\epsilon_1) = \begin{cases} 
\Pi_1^b(x_1,y_1,p_1,r_1,\epsilon_1) & \epsilon_1 < y_1 - E[D] \\
\Pi_1^h(x_1,y_1,p_1,r_1,\epsilon_1) & \epsilon_1 \geq y_1 - E[D]
\end{cases}
\]

\[
= \begin{cases} 
p_1(E[D] + \epsilon_1) - c(y_1 - x_1) - h(y_1 - E[D] - \epsilon_1) & \epsilon_1 < y_1 - E[D] \\
p_1(E[D] + \epsilon_1) - c(y_1 - x_1) - b(E[D] + \epsilon_1 - y_1) & \epsilon_1 \geq y_1 - E[D].
\end{cases} \tag{5.2.4}
\]

Similarly we distinguish between the possible realizations of \( v_2(x_2(y_1,p_1,r_1),r_2(p_1,r_1)) \):

\[
v_2(x_2(y_1,p_1,r_1),r_2(p_1,r_1)) = \begin{cases} 
V_2^*(r_2(p_1,r_1)) & \epsilon_1 \geq y_1 - E[D] - S^*(r_2) \\
\Psi_2^*(x_2(y_1,p_1,r_1,\epsilon_1),r_2(p_1,r_1)) & \epsilon_1 < y_1 - E[D] - S^*(r_2)
\end{cases}
\]

\[
= \begin{cases} 
J_2(x_2,S^*(r_2),P_2^*(r_2),r_2) & \epsilon_1 \geq y_1 - E[D] - S^*(r_2) \\
J_2(x_2,S^*(r_2),P_2^*(r_2),r_2) & \epsilon_1 < y_1 - E[D] - S^*(r_2).
\end{cases} \tag{5.2.5}
\]

We now exchange the order of summation and taking expected values such that

\[
J_1(x_1,y_1,p_1,r_1) = E[\Pi_1(x_1,y_1,p_1,r_1,\epsilon_1) + \gamma v_2(x_2(y_1,p_1,r_1,\epsilon_1),r_2(p_1,r_1))] =
\]

\[
\int_{u_1} \left[ \begin{cases} 
\Pi_1^b(x_1,y_1,p_1,r_1,u_1) & u_1 < y_1 - E[D] \\
\Pi_1^h(x_1,y_1,p_1,r_1,u_1) & u_1 \geq y_1 - E[D]
\end{cases} \right] f(u_1)du_1 +
\]

\[
+ \gamma \left[ \begin{cases} 
V_2^*(r_2(p_1,r_1)) & u_1 \geq y_1 - E[D] - S^*(r_2) \\
\Psi_2^*(x_2(y_1,p_1,r_1,u_1),r_2(p_1,r_1)) & u_1 < y_1 - E[D] - S^*(r_2)
\end{cases} \right] f(u_1)du_1. \tag{5.2.6}
\]

From the formula above it is easy to see that we need to distinguish between the three cases \( S^*(r_2) > 0 \), \( S^*(r_2) = 0 \) and \( S^*(r_2) < 0 \) (see figure 5.5). Let \( S^*(r_2) > 0 \), then \( J_1(x_1,y_1,p_1,r_1) \) becomes

\[
J_1(x_1,y_1,p_1,r_1) = \\
= \int_{-\infty}^{y_1 - E[D] - S^*(r_2)} \left( \Pi_1^b(x_1,y_1,p_1,r_1,u_1) + \gamma \Psi_2^*(x_2(y_1,p_1,r_1,u_1),r_2(p_1,r_1)) \right) f(u_1)du_1 \\
+ \int_{y_1 - E[D] - S^*(r_2)}^{y_1 - E[D]} \left( \Pi_1^h(x_1,y_1,p_1,r_1,u_1) + \gamma V_2^*(r_2(p_1,r_1)) \right) f(u_1)du_1 \\
+ \int_{y_1 - E[D]}^{\infty} \left( \Pi_1^h(x_1,y_1,p_1,r_1,u_1) + \gamma V_2^*(r_2(p_1,r_1)) \right) f(u_1)du_1. \tag{5.2.7}
\]
For $S^*(r_2) = 0$ the profit function $J_1(x_1, y_1, p_1, r_1)$ can be written as

$$J_1(x_1, y_1, p_1, r_1) =$$

$$= \int_{\gamma_1 - E[D]}^{\gamma_1 - E[D] - S^*(r_2)} \left( \Pi^1_1(x_1, y_1, p_1, r_1, u_1) + \gamma \Psi^2_2(x_2(y_1, p_1, r_1, u_1), r_2(p_1, r_1)) \right) f(u_1) du_1$$

$$+ \int_{\gamma_1 - E[D] - S^*(r_2)}^{\gamma_1 - E[D]} \left( \Pi^1_1(x_1, y_1, p_1, r_1, u_1) + \gamma V^*_2(r_2(p_1, r_1)) \right) f(u_1) du_1.$$

(5.2.8)

We now consider the case $S^*(r_2) < 0$, which yields

$$J_1(x_1, y_1, p_1, r_1) =$$

$$= \int_{\gamma_1 - E[D]}^{\gamma_1 - E[D] - S^*(r_2)} \left( \Pi^1_1(x_1, y_1, p_1, r_1, u_1) + \gamma \Psi^2_2(x_2(y_1, p_1, r_1, u_1), r_2(p_1, r_1)) \right) f(u_1) du_1$$

$$+ \int_{\gamma_1 - E[D] - S^*(r_2)}^{\gamma_1 - E[D]} \left( \Pi^1_1(x_1, y_1, p_1, r_1, u_1) + \gamma V^*_2(r_2(p_1, r_1)) \right) f(u_1) du_1.$$

(5.2.9)

We will now show that the four possible summands are jointly concave in $y_1$ and $p_1$.

**Lemma 5.5.** For the linear demand function defined in equation (5.0.1), the system (4.3.1) to (4.3.3) and a two-period setting $T = 2$ each of the functions

$$\Pi^1_1(x_1, y_1, p_1, r_1, \epsilon_1) + \gamma V^*_2(r_2(p_1, r_1)),$$

(5.2.10)

$$\Pi^0_1(x_1, y_1, p_1, r_1, \epsilon_1) + \gamma V^*_2(r_2(p_1, r_1)),$$

(5.2.11)

$$\Pi^1_1(x_1, y_1, p_1, r_1, \epsilon_1) + \gamma \Psi^2_2(x_2(y_1, p_1, r_1, \epsilon_1), r_2(p_1, r_1)),$$

(5.2.12)

$$\Pi^0_1(x_1, y_1, p_1, r_1, \epsilon_1) + \gamma \Psi^2_2(x_2(y_1, p_1, r_1, \epsilon_1), r_2(p_1, r_1))$$

(5.2.13)

is jointly concave in $y_1$ and $p_1$.

**Proof.** The first two functions are trivially jointly concave in $y_1$ and $p_1$, since the two possible realizations of the profit function in the first time-period $\Pi^1_1(x_1, y_1, p_1, r_1, \epsilon_1)$ and $\Pi^0_1(x_1, y_1, p_1, r_1, \epsilon_1)$ are jointly concave by lemma A.1 in appendix A and $V^*_2(r_2(p_1, r_1))$ is jointly concave by lemma A.2 with a discount factor $\gamma > 0$.

The situation is not so clear for the second two functions, since the expected optimal profit $\Psi^2_2(x_2(y_1, p_1, r_1, \epsilon_1), r_2(p_1, r_1))$ is not necessarily concave in $p_1$ by equation (A.10) in lemma A.3 and thus $\Psi^*_2(x_2(y_1, p_1, r_1, \epsilon_1), r_2(p_1, r_1))$ is not jointly concave in $y_1$ and $p_1$. However, in the following we will show that the joint concavity of the first time period's profits $\Pi_1(x_1, y_1, p_1, r_1, \epsilon_1)$ is strong enough to dominate the non-concavity of the future
profit. It is easy to see that the 'misbehavior' of non-concavity will be worse, the larger the discount factor \( \gamma \) is. Hence, without loss of generality, we examine the case of no discount \( (\gamma = 1) \) in the following analysis.

As a first step we will now show that \( \Pi_1(x_1, y_1, p_1, r_1, \epsilon_1) + \mathcal{D}_2(x_2, r_2) \) is concave in the selling price \( p_1 \). By using lemma A.1 and lemma A.3 from appendix A, as well as the assumptions that \( \beta_1, \beta_2 \leq 0 \) and \( 0 \leq \alpha < 1 \), we can calculate the following for linear demand function (5.0.1):

\[
\frac{\partial^2}{\partial p_1^2} \left( \Pi_1(x_1, y_1, p_1, r_1, \epsilon_1) + \mathcal{D}_2(x_2, y_1, p_1, r_1, \epsilon_1), r_2(p_1, r_1) \right) =
\frac{-2(2\beta_1 + (1 - \alpha)\beta_3)((h + b)(\beta_1 - \beta_2)^2 + ((\alpha - 3)\beta_2 - 2\beta_3)/(2f(x_2 - E[D(p_2, r_2)]))}{(\beta_1 + \beta_2)(2/f(x_2 - E[D(p_2, r_2)]) - (h + b)(\beta_1 + \beta_2))}.
\]

(5.2.14)

which is less than or equal to zero and thus proves concavity in \( p_1 \). It remains to show joint concavity in \( y_1 \) and \( p_1 \). For this purpose we evaluate, by again using lemmas A.1 and A.3, the determinant of the Hesse matrix of \( \Pi_1(x_1, y_1, p_1, r_1, \epsilon_1) + \mathcal{D}_2(x_2, y_1, p_1, r_1, \epsilon_1), r_2(p_1, r_1) \) for the linear demand function (5.0.1), where \( H[\cdot] \) denotes the Hesse matrix:

\[
\begin{align*}
\det \left( H \left[ \Pi_1(x_1, y_1, p_1, r_1, \epsilon_1) \right] + H \left[ \mathcal{D}_2(x_2, y_1, p_1, r_1, \epsilon_1), r_2(p_1, r_1) \right] \right) &= \\
\left( \frac{\partial^2 \Pi_1(x_1, y_1, p_1, r_1, \epsilon_1)}{\partial p_1^2} + \frac{\partial^2 \mathcal{D}_2(x_2, y_1, p_1, r_1, \epsilon_1), r_2(p_1, r_1)}{\partial p_1^2} \right) \\
\left( \frac{\partial^2 \Pi_1(x_1, y_1, p_1, r_1, \epsilon_1)}{\partial y_1^2} + \frac{\partial^2 \mathcal{D}_2(x_2, y_1, p_1, r_1, \epsilon_1), r_2(p_1, r_1)}{\partial y_1^2} \right) \\
\left( \frac{\partial^2 \Pi_1(x_1, y_1, p_1, r_1, \epsilon_1)}{\partial y_1 \partial p_1} + \frac{\partial^2 \mathcal{D}_2(x_2, y_1, p_1, r_1, \epsilon_1), r_2(p_1, r_1)}{\partial y_1 \partial p_1} \right)^2 &= \\
\frac{(8(\beta_1 + \beta_2)^2 - (1 - \alpha)^2(\beta_3^2)(h + b)}{\left( (\beta_1 + \beta_2)(2/f(x_2 - E[D(p_2, r_2)]) - (h + b)(\beta_1 + \beta_2) \right) \geq 0.
\end{align*}
\]

(5.2.15)

The above equation shows that the Hesse matrix of \( \Pi_1(x_1, y_1, p_1, r_1, \epsilon_1) + \mathcal{D}_2(x_2, r_2) \) is positive semi-definite, which yields joint concavity in the decision variables \( y_1 \) and \( p_1 \) and proves the lemma.

It now remains to be shown that the profit functions of the first and second time-period, respectively, are continuous and concave in their points of non-differentiability.

**Lemma 5.6.** The two functions \( \Pi_1(x_1, y_1, p_1, r_1, \epsilon_1) \) and \( \mathcal{D}_2(x_2, y_1, p_1, r_1, \epsilon_1), r_2(p_1, r_1) \) defined piecewise in equations (5.2.4) and (5.2.5) are continuous and concave in \( y_1 \) and \( p_1 \) in their points of non-differentiability.
Proof. For proving the above lemma 5.6 it suffices to show the following properties for any value of \( \epsilon_1 = y_1 - E[D(p_1, r_1, \epsilon_1)] \):

\[
\lim_{\delta \to 0} \left[ \Pi_1(x_1, y_1, p_1 - \delta, r_1, \epsilon_1) - \Pi_1(x_1, y_1, p_1 + \delta, r_1, \epsilon_1) \right] = 0,
\]

\[
\lim_{\delta \to 0} \left[ \Pi_1(x_1, y_1 - \delta, p_1, r_1, \epsilon_1) - \Pi_1(x_1, y_1 + \delta, p_1, r_1, \epsilon_1) \right] = 0,
\] (5.2.16)

\[
\lim_{\delta \to 0} \left[ \frac{\partial \Pi_1(x_1, y_1, p_1 - \delta, r_1, \epsilon_1)}{\partial p_1} - \frac{\partial \Pi_1(x_1, y_1, p_1 + \delta, r_1, \epsilon_1)}{\partial p_1} \right] \geq 0,
\]

\[
\lim_{\delta \to 0} \left[ \frac{\partial \Pi_1(x_1, y_1 - \delta, p_1, r_1, \epsilon_1)}{\partial y_1} - \frac{\partial \Pi_1(x_1, y_1 - \delta, p_1, r_1, \epsilon_1)}{\partial y_1} \right] \geq 0. \] (5.2.17)

Furthermore, we need to show for any value of \( \epsilon_1 = y_1 - E[D(p_1, r_1, \epsilon_1)] - S^*(r_2(p_1, r_1)) \) that the following equations hold true:

\[
\lim_{\delta \to 0} \left[ v_2(x_2(y_1, p_1 - \delta, r_1, \epsilon_1), r_2(p_1 - \delta, r_1) - v_2(x_2(y_1, p_1 + \delta, r_1, \epsilon_1), r_2(p_1 + \delta, r_1)) \right] = 0,
\]

\[
\lim_{\delta \to 0} \left[ v_2(x_2(y_1 - \delta, p_1, r_1, \epsilon_1), r_2(p_1, r_1) - v_2(x_2(y_1 + \delta, p_1, r_1, \epsilon_1), r_2(p_1, r_1)) \right] = 0,
\] (5.2.18)

\[
\lim_{\delta \to 0} \left[ \frac{\partial v_2(x_2(y_1, p_1 - \delta, r_1, \epsilon_1), r_2(p_1 - \delta, r_1))}{\partial p_1} - \frac{\partial v_2(x_2(y_1, p_1 + \delta, r_1, \epsilon_1), r_2(p_1 + \delta, r_1))}{\partial p_1} \right] = 0,
\]

\[
\lim_{\delta \to 0} \left[ \frac{\partial v_2(x_2(y_1 - \delta, p_1, r_1, \epsilon_1), r_2(p_1, r_1))}{\partial y_1} - \frac{\partial v_2(x_2(y_1 + \delta, p_1, r_1, \epsilon_1), r_2(p_1, r_1))}{\partial y_1} \right] = 0. \] (5.2.19)

Equations (5.2.16) and (5.2.18) show the functions' continuity and equations (5.2.17) and (5.2.19) the functions' concavity in \( y_1 \) and \( p_1 \). Lemma A.5 and lemma A.6 in appendix A shows that the above equations indeed hold.

We will now show that the function \( \Pi_1(x_1, y_1, p_1, r_1, \epsilon_1) \) is jointly concave in \( y_1 \) and \( p_1 \) at the kink \( \epsilon_1 = y_1 - E[D(p_1, r_1, \epsilon_1)] \).

**Lemma 5.7.** The profit function of the first time period \( \Pi_1(x_1, y_1, p_1, r_1, \epsilon_1) \) defined piecewise in equation (5.2.4) is jointly concave in \( y_1 \) and \( p_1 \).

**Proof.** We know by lemma A.1 in appendix A that the functions \( \Pi_1^h(x_1, y_1, p_1, r_1, \epsilon_1) \) and \( \Pi_1^b(x_1, y_1, p_1, r_1, \epsilon_1) \) are both jointly concave in \( y_1 \) and \( p_1 \). It now remains to be shown that \( \Pi_1(x_1, y_1, p_1, r_1, \epsilon_1) \) is jointly concave at the kink \( \epsilon_1 = y_1 - E[D(p_1, r_1, \epsilon_1)] \). We for convenience reformulate equation (5.2.4) in the following way:

\[
\Pi_1(x_1, y_1, p_1, r_1, \epsilon_1) = p_1(E[D(p_1, r_1, \epsilon_1)] + \epsilon_1) - c(y_1 - x_1) - q(y_1, p_1, r_1, \epsilon_1), \] (5.2.20)
with

\[
g(y_1, p_1, r_1, \epsilon_1) = \begin{cases}
    h(y_1 - E[D(p_1, r_1, \epsilon_1)] - \epsilon_1), & \epsilon_1 < y_1 - E[D(p_1, r_1, \epsilon_1)] \\
    -b(y_1 - E[D(p_1, r_1, \epsilon_1)] - \epsilon_1), & \epsilon_1 \geq y_1 - E[D(p_1, r_1, \epsilon_1)].
\end{cases}
\]

(5.2.21)

It is clear that \( p_1(E[D(p_1, r_1, \epsilon_1)] + \epsilon_1) - c(y_1 - x_1) \) is jointly concave in \( y_1 \) and \( p_1 \). Let \( \{ (\tilde{y}_1, \tilde{p}_1) : \tilde{y}_1 = E[D(\tilde{p}_1, r_1, \epsilon_1)] + \epsilon_1 \} \) and \( \{ (\hat{y}_1, \hat{p}_1) : \hat{y}_1 \neq E[D(\hat{p}_1, r_1, \epsilon_1)] + \epsilon_1 \} \). It is easy to see from equation (5.2.21) and \( b, h \geq 0 \) that

\[
g(\tilde{y}_1, \tilde{p}_1, r_1, \epsilon_1) < g(\hat{y}_1, \hat{p}_1, r_1, \epsilon_1).
\]

(5.2.22)

Since \( y_1 - E[D(p_1, r_1, \epsilon_1)] - \epsilon_1 \) is linear in \( y_1 \) it follows directly that \( g(y_1, p_1, r_1, \epsilon_1) \) is jointly convex and thus \( \Pi_1(x_1, y_1, p_1, r_1, \epsilon_1) \) jointly concave in \( y_1 \) and \( p_1 \) at the kink \( \epsilon_1 = y_1 - E[D(p_1, r_1, \epsilon_1)] \) and therefore anywhere.

We are now ready to show the optimality of a base-stock-list-price policy, for which we introduce another useful lemma, which can be found in Heuser (1981).

**Lemma 5.8** (Heuser (1981), theorem 166.1). Let the function \( f : G \subset \mathbb{R}^n \rightarrow \mathbb{R} \) (\( G \) being an open set) be differentiable in \( G \). Then the derivative in direction \( a = (a_1, \ldots, a_n) \) exists and is given by

\[
\frac{\partial f(x)}{\partial a} = \sum_{i=1}^{n} a_i \frac{\partial f(x)}{\partial x_i}.
\]

(5.2.23)

**Theorem 5.3** (Base-stock-list-price policy). For the linear demand function defined in equation (5.0.1) and the system (4.3.1) to (4.3.3), the following holds for \( t = 1, 2 \) of a two-period setting \( T = 2 \):

1. A base-stock-list-price policy is optimal.
2. The profit function \( J_t(y_t, p_t, x_t, r_t) \) is jointly concave in \( y_t \) and \( p_t \).
3. The profit function \( J_t(y_t, p_t, x_t, r_t) \) is submodular in \( y_t \) and \( p_t \).

**Proof.** The above statement is true for the second time period \( t = 2 \) by theorem 5.1. It remains to be shown that \( J_1(x_1, y_1, p_1, r_1) \) is jointly concave and submodular in \( y_1 \) and \( p_1 \).

In lemma 5.5 we showed that the functions defined in equations (5.2.10) to (5.2.13) are each jointly concave in \( y_1 \) and \( p_1 \). We now need to show that the junctions of those functions are indeed jointly concave. It is convenient that we already know from equation (A.20) and (A.21) in lemma A.6 in appendix A that when \( V^*_2(x_2(y_1, p_1, r_1, \epsilon_1), r_2(p_1, r_1)) \) switches to \( \Omega_2^*(x_2(y_1, p_1, r_1, \epsilon_1), r_2(p_1, r_1)) \), the difference of its slopes with respect to \( y_1 \) and \( p_1 \) is zero and thus by \( v_2(x_2(y_1, p_1, r_1, \epsilon_1), r_2(p_1, r_1)) \) being continuous, it is also differentiable

\[1\text{A function } f \text{ is differentiable in } G \subset \mathbb{R}^n \text{, if the partial derivatives of } f \text{ with respect to } x_1, \ldots, x_n \text{ exist for any point } \bar{x} \in G. \]
with respect to $y_1$ and $p_1$. By Lemma 5.8 it follows that $v_2(x_2(y_1, p_1, r_1, \epsilon_1), r_2(p_1, r_1))$ is differentiable in any direction $a = a_1 y_1 + a_2 p_1$ and thus is a smooth function. Therefore the problem of showing that the junctions of (5.2.10) to (5.2.13) are jointly concave reduces to the verification that the junction of the two piecewise functions

$$
\Pi^b_1(x_1, y_1, p_1, r_1, \epsilon_1) + \gamma v_2(x_2(y_1, p_1, r_1, \epsilon_1), r_2(p_1, r_1))
$$

(5.2.24)

and

$$
\Pi^b_1(x_1, y_1, p_1, r_1, \epsilon_1) + \gamma v_2(x_2(y_1, p_1, r_1, \epsilon_1), r_2(p_1, r_1))
$$

(5.2.25)

is indeed jointly concave, which follows from lemma 5.7.

It now remains to be shown that $J_1(x_1, y_1, p_1, r_1)$ is submodular in $y_1$ and $p_1$ and thus $\partial^2 J_1(x_1, y_1, p_1, r_1)/(\partial y_1 \partial p_1) \leq 0$. Since we have $\partial^2 \Pi_1(x_1, y_1, p_1, r_1, \epsilon_1)/(\partial y_1 \partial p_1) = 0$ by equation (A.3) in lemma A.1, $\partial^2 V^*_2(r_2(p_1, r_1))/(\partial y_1 \partial p_1) = 0$ by equation (A.7) in lemma A.2 and $\partial^2 G_1(y_1, p_1, r_1)/(\partial y_1 \partial p_1) \leq 0$ by equation (A.12) in lemma A.3, and submodularity is maintained by integration it is clear that $J_1(x_1, y_1, p_1, r_1) = E[\Pi_1(x_1, y_1, p_1, r_1, \epsilon_1)] + \gamma v_2(x_2(y_1, p_1, r_1, \epsilon_1), r_2(p_1, r_1))$ is submodular in $y_1$ and $p_1$. By theorem 8-4 in Heyman and Sobel (2004) submodularity in $y_1$ and $p_1$ suffices to show that the optimal price $p^*_1(x_1, r_1)$ is non-increasing in $x_1$ and a list-price policy is optimal. \qed

5.3. Multi-period model

Under some restrictive assumptions we are able to extend the above attained base-stock property to the multi-period-case. For the ease of proving we now consider the transformed model given by equations (4.3.7) to (4.3.9) in chapter 4. Furthermore, we introduce the following assumptions:

**Assumption 5.1.** In each time period $t = 1, \ldots, T$ the following holds: The demand function $D_t(p, r, \epsilon)$ is non-increasing in $p$, non-decreasing in $r$ and jointly concave in $p$ and $r$, while the revenues $pD_t(p, r, \epsilon)$ are assumed to be jointly concave in $p$ and $r$. Furthermore, $G_t(y, p, r)$ is assumed to be jointly convex in $y$, $p$ and $r$.

**Lemma 5.9.** The expected profit-to-go function $V_t(r, x)$ is non-increasing in $x$ for all $r$ and $t = 1, \ldots, T$.

**Proof.** Let $x_1 < x_2$. Then $V_t(r, x_1) = \max_{y \geq x_1} J_t(y, p, r) \geq \max_{y \geq x_2} J_t(y, p, r) = V_t(r, x_2)$. \qed

**Theorem 5.4 (Base-stock policy).** For the system (4.3.7) to (4.3.9) and assumption 5.1, the following holds for any time period $t = 1, \ldots, T$:

1. $J_t(y, p, r)$ is jointly concave in $y, p$ and $r$.
2. $V_t(x, r)$ is jointly concave in $x$ and $r$.
3. A base-stock policy with order-up-to level $S_t^*(x, r)$ is optimal in time period $t$. \qed
5.3. MULTI-PERIOD MODEL

Proof. The proof technique is similar to the one suggested in theorem 2.5 and is conducted by induction. \( V_{T+1}(x, r) = 0 \) is trivially jointly concave in \( x \) and \( r \) and thus \( J_T(y, p, r) \) is jointly concave in \( y, p \) and \( r \) by assumption 5.1. We now assume that \( V_{t+1}(x, r) \) is jointly concave in \( x \) and \( r \) and \( J_{t+1}(y, p, r) \) is jointly concave in \( y, p \) and \( r \).

By defining \( \tilde{r}(p, r) \) and \( \tilde{x}(y, p, r, \epsilon) \) as

\[
\tilde{r}(p, r) = \alpha r + (1 - \alpha)p, \\
\tilde{x}(y, p, r, \epsilon) = y - D_t(p, r, \epsilon),
\]

the following holds for any pair \((y_1, y_2), (p_1, p_2)\) and \((r_1, r_2)\):

\[
\tilde{x}\left(\frac{y_1 + y_2}{2}, \frac{p_1 + p_2}{2}, \frac{r_1 + r_2}{2}, \epsilon\right) \leq \frac{y_1 + y_2}{2} - D_t\left(\frac{p_1 + p_2}{2}, \frac{r_1 + r_2}{2}, \epsilon\right) = \frac{1}{2}(y_1 - D_t(p_1, r_1, \epsilon)) + \frac{1}{2}(y_2 - D_t(p_2, r_2, \epsilon)) = \frac{1}{2}\tilde{x}(y_1, p_1, r_1, \epsilon) + \frac{1}{2}\tilde{x}(y_2, p_2, r_2, \epsilon).
\]

since \( D_t(p, r, \epsilon) \) is jointly concave in \( p \) and \( r \) by assumption 5.1. Thus we obtain

\[
V_{t+1}\left(\tilde{x}\left(\frac{y_1 + y_2}{2}, \frac{p_1 + p_2}{2}, \frac{r_1 + r_2}{2}, \epsilon\right), \tilde{r}\left(\frac{p_1 + p_2}{2}, \frac{r_1 + r_2}{2}\right)\right) \geq V_{t+1}\left(\frac{1}{2}\tilde{x}(y_1, p_1, r_1, \epsilon) + \frac{1}{2}\tilde{x}(y_2, p_2, r_2, \epsilon), \tilde{r}(p_1, r_1) + \frac{1}{2}\tilde{r}(p_2, r_2)\right) \geq V_{t+1}\left(\frac{1}{2}\tilde{x}(y_1, p_1, r_1, \epsilon), \tilde{r}(p_1, r_1)\right) + \frac{1}{2}V_{t+1}\left(\tilde{x}(y_2, p_2, r_2, \epsilon), \tilde{r}(p_2, r_2)\right).
\]

The first inequality of (5.3.4) holds, due to Lemma 5.9 and the following equality since \( \tilde{r}\left(\frac{p_1 + p_2}{2}, \frac{r_1 + r_2}{2}\right) = \frac{1}{2}\tilde{r}(p_1, r_1) + \frac{1}{2}\tilde{r}(p_2, r_2) \). The second inequality of (5.3.4) follows from \( V_{t+1}(x, r) \) being jointly concave in \( x \) and \( r \) by the induction assumption. It is clear that equation (5.3.4) guarantees that \( V_{t+1}(\tilde{x}(y, p, r, \epsilon), \tilde{r}(p, r)) \) and thus \( E[V_{t+1}(x, r)] \) is jointly concave in \( p, y \) and \( r \). Since the first three terms of (4.3.8) are jointly concave in \( y, p \) and \( r \) by assumption 5.1, \( J_t(y, p, r) \) is jointly concave in \( y, p \) and \( r \).

We now show that \( V_t(x, r) \) is jointly concave in \( x \) and \( r \). From (4.3.7) we know that

\[
V_t\left(\frac{x_1 + x_2}{2}, \frac{r_1 + r_2}{2}\right) = \max_{y \geq \frac{r_1 + r_2}{2}} \left\{ J_t(y, p, \frac{r_1 + r_2}{2}) \right\}.
\]

Since \( J_t(y, p, r) \) is jointly concave in \( y \) and \( p \) a base-stock policy with a base-stock level \( S^*(r) \) and an optimal price \( p^*(x, r) \) is optimal and thus the optimal profit \( V_t(x, r) \) can be
written as

\[ V_t(x, r) = J_t(S^*(r), p^*(x, r), r). \quad (5.3.6) \]

It furthermore follows that

\[
V_t \left( \frac{x_1 + x_2}{2}, \frac{r_1 + r_2}{2} \right) = \max_{y \geq \frac{x_1 + x_2}{2}, p} \left\{ J_t \left( y, p, \frac{r_1 + r_2}{2} \right) \right\} = \\
J_t \left( \max \left( S^* \left( \frac{r_1 + r_2}{2}, \frac{x_1 + x_2}{2} \right), \frac{x_1 + x_2}{2} \right), p^* \left( \frac{x_1 + x_2}{2}, \frac{r_1 + r_2}{2} \right) \right) \geq \\
J_t \left( \max \left( S^* (r_1), x_1 \right), \max \left( S^* (r_2), x_2 \right), p^*(x_1, r_1), p^*(x_2, r_2) \right) \right) \geq \\
J_t \left( \max \left( S^* (r_1), x_1 \right), p^*(x_1, r_1), r_1 \right) + J_t \left( \max \left( S^* (r_2), x_2 \right), p^*(x_2, r_2), r_2 \right) \right) = \\
\frac{V_t \left( x_1, r_1 \right)}{2} + \frac{V_t \left( x_2, r_2 \right)}{2}. \quad (5.3.7) \]

The first inequality of (5.3.7) holds since \( p^* \left( \frac{x_1 + x_2}{2}, \frac{r_1 + r_2}{2} \right) \) and \( \max \left( S^* \left( \frac{r_1 + r_2}{2}, \frac{x_1 + x_2}{2} \right) \right) \) are the global optima of \( V_t \left( \frac{x_1 + x_2}{2}, \frac{r_1 + r_2}{2} \right) \). Any other solution, particularly \( p^* (x_1, r_1) + p^* (x_2, r_2) \) and \( \max(S^*(r_1, x_1) + \max(S^*(r_2, x_2)) \) will thus be less or equal to the optimal solution. \( J_t(y, p, r) \) being jointly concave in \( y, p \) and \( r \) explains the second inequality of (5.3.7). \( \square \)

Remark 5.4. Note that for the linear demand function defined in equation (5.0.1) the revenues \( pD_1(p, r, \epsilon) \) are not jointly concave in \( p \) and \( r \) and thus assumption 5.1 does not hold.

Similar to sections 2.3.3 and 3.3 we can find a possible steady state for the integrated pricing and inventory control model given by equations (4.3.1) to (4.3.3).

**Theorem 5.5.** If the system (4.3.1) to (4.3.3) admits a steady state, then for the linear demand function defined in equation (5.0.1) and the loss-neutral case (\( \beta_2 = \beta_3 \)) it is given by

\[
p^*_\infty = \frac{(\beta_1 c - \beta_3)(1 - \alpha \gamma) + \beta_2 (1 - \gamma) c}{2 \beta_1 (1 - \alpha \gamma) + \beta_2 (1 - \gamma)}, \quad (5.3.8)\]

\[
y^*_\infty = E[D(p^*_\infty, p^*_\infty, \epsilon)] + F^{-1} \left( \frac{b - (1 - \gamma) c}{h + b} \right). \quad (5.3.9)\]

**Proof.** The proof is a combination of the proofs of theorem 2.9 in section 2.3.3 and theorem 3.1 in section 3.3. As in the proof of the joint pricing and inventory without reference effect case, we first find the steady state inventory. Again the optimal total discounted profit for an
infinite time horizon is given by (compare equation (4.2.2)):

\[ V(x_1, r_1) = cx_1 + \sum_{t=0}^{\infty} \gamma^t \max_{y_{t+1}, r_{t+1}} [p_{t+1} E[D(p_{t+1}, r_{t+1}, \epsilon_{t+1})] - c(1 - \gamma)y_{t+1} + \gamma cD(p_{t+1}, r_{t+1}, \epsilon_{t+1}) - G(y_{t+1}, p_{t+1}, r_{t+1})] \]

where

\[ G(y_{t+1}, p_{t+1}, r_{t+1}) = h \int_{-\infty}^{y_{t+1} - E[D(p_{t+1}, r_{t+1}, \epsilon_{t+1})]} (y_{t+1} - E[D(p_{t+1}, r_{t+1}, \epsilon_{t+1})] - u)f(u)du + b \int_{y_{t+1} - E[D(p_{t+1}, r_{t+1}, \epsilon_{t+1})]}^{\infty} (u - y_{t+1} - E[D(p_{t+1}, r_{t+1}, \epsilon_{t+1})])f(u)du. \]

Differentiation with respect to \( y \) and using the same arguments as in the proof of theorem 2.9, we get the steady state base-stock equation (5.3.9) depending on the price. Note that the reference price has to be equal to the price as this is necessary for the steady state price. The steady state price in formula (5.3.8) is found analogously to the proof of the steady state for the reference price model (see theorem 3.1) after substituting \( y_\infty^*(p) \) for \( y \) in the infinite sum (5.3.10) and differentiating with respect to the price \( p \). This yields exactly the same steady state price \( p_\infty^* \) as in theorem 3.1.

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**Table 5.1.: Steady-state base stock and list price**

**Remark 5.5.** Note that the optimal steady state price \( p_\infty^* \) in equation (5.3.8) is the same as in theorem 2.9 in section 2.3.3 and the optimal steady state base-stock level \( y_\infty^* \) in equation (5.3.9) corresponds to the optimal steady state price obtained in theorem 3.1 in section 3.3.
As in section 2.3.3 and 3.3 one can study the behavior of the steady state solutions (see table 5.1) for the integrated model under consideration in this chapter.

**Remark 5.6.** From remark 5.5 it is clear that a sensitivity analysis of the steady state price $p^*_\infty$ of equation (5.3.8) is identical to the one described in theorem 3.3 in section 3.3 and thus the optimal steady state price $p^*_\infty$ is decreasing in the memory parameter $\alpha$, increasing in the discount factor $\gamma$ and decreasing in the reference effect $|\beta_2|$.

Since $F^{-1}\left(\frac{b-(1-\gamma)\epsilon}{h+b}\right)$ is independent of both $\alpha$ and $|\beta_2|$, $p^*_\infty$ is decreasing in $\alpha$ and $|\beta_2|$ and expected demand is decreasing in price, it is clear that $E[D(p^*_\infty, p^*_\infty, \epsilon)]$ is increasing in $\alpha$ and $|\beta_2|$. As a consequence, $y^*_\infty$ is increasing in $\alpha$ and $|\beta_2|$. However, for the discount factor $\gamma$, we cannot make a definite statement about the behavior of the steady state base-stock $y^*_\infty$, as the safety stock is increasing in $\gamma$ while $E[D(p^*_\infty, p^*_\infty, \epsilon)]$ is decreasing in $\gamma$. As a result, in case of small uncertainty in demand the base-stock is decreasing in $\gamma$ whereas for large uncertainty the base-stock is increasing in $\gamma$. A numerical example is given in table 5.1.