Chapter 2

A Review of the Newsvendor Model

The newsvendor model is a single period inventory control model. The product in question is sold during one season and the ordering (or production) quantity should be set before the season starts. The demand during the selling period is not known before the season starts. Thus the ordering decision is made under uncertainty about the amount of demand. During the selling season, it is not possible to order additional units and cover the unexpected part of demand. At the end of the season unsatisfied demand is lost and the leftover inventory is obsolete.

One of the application areas of the newsvendor model is the inventory management of perishable goods such as fresh produce or newspapers. These products are naturally single period products since they have a limited useful life. Another area is the fast-changing markets with short life-cycle products such as fashion textiles (Fisher and Raman (1996)) or consumer electronics. In these industries the production and transportation lead-times are generally longer than the market lifetime of the product. In this case, it is not possible to order additional units during the selling season. Additionally, the newsvendor model is relevant to the capacity management and revenue management problems (see e.g. Van Mieghem and Rudi (2002), McGill and van Ryzin (1999)), because of the one-time irreversible nature of the decisions and the stochastic environment in these problems.

Other than its applicability, the newsvendor model provides important structural results which are not possible to derive in the multi-period setting because of the models’ complexity. These results help better understand the multi-period problems since the newsvendor model is the building block of the multi-period models. From a technical point of view, the analytical treatment of the multi-period models often include dynamic programming models that rely on inductive proofs. The properties of the single period problem constitute the starting point for the analysis. Moreover, when infinite horizon problems are considered, the stationary policies often turn out to share the same characteristics with the single period solutions.

Traditionally, the newsvendor problem deals only with the inventory decision. The selling price and the corresponding demand forecast are considered as input parameters for the newsvendor’s inventory decision.
In this situation the newsvendor acts as a price-taker. However, in many situations the newsvendor has a pricing power on his product. The fashion retailers or suppliers of popular electronics products can set the price of their products and affect their demand. The class of price-setting newsvendors is the subject of this research.

2.1 Price-taking newsvendor model

For the price-taking newsvendor, the price of the product is an exogenous variable. It can be assumed that the newsvendor is a small player in a perfectly competitive market and has no power to influence his selling price or he is a retailer of a product for which the price is set centrally. Given that the selling price is $p$ and the purchasing cost per unit is $c$, he has to decide on the ordering quantity $y$.

Demand, $X$, is a random variable which is realized after the ordering decision. It follows a known probability distribution with a strictly increasing distribution function $F(\cdot)$, which is independent of price, and the corresponding density function $f(\cdot)$. If demand during the period turns out to be less than the ordering quantity the newsvendor ends up with some leftover inventory which can not be carried to another selling season. On the other hand, if demand is larger than the ordering quantity the unsatisfied demand is lost since there is no further replenishment option. Since there is a single selling period it is generally assumed that before the ordering decision the newsvendor does not have any initial inventory, though the positive initial inventory requires just a simple modification of the basic model.

Since demand is a random variable, the resulting profit is also a random variable which depends on the ordering quantity. If $y$ is the ordering quantity and $x$ is the demand realization, profit is

\[
\text{Profit} = \begin{cases} 
px - cy & x \leq y \\
py - cy & x > y 
\end{cases}
\]

The objective of the newsvendor is to find the optimum ordering quantity $y^*$ which maximizes the expected profit $\Pi(y)$ where

\[
\Pi(y) = \int_0^y [px - cy]f(x)dx + \int_y^\infty [py - cy]f(x)dx \\
= (p - c)y - p \int_0^y F(x)dx.
\]
The first derivative of the expected profit function is

\[ \Pi_y(y) = (p - c) - pF(y). \]

If we set \( \Pi_y(y) \) equal to zero and solve for \( y \) we derive the optimum ordering quantity \( y^* \) as:

\[ y^* = F^{-1}\left(\frac{p - c}{p}\right) \tag{2.1} \]

where \( F^{-1} \) is the inverse distribution function of demand.

\( y^* \) is the unique maximizer of expected profit since \( \Pi(y) \) is strictly concave as can be observed from the negativity of the second derivative:

\[ \Pi_{yy}(y) = -pf(y) < 0. \]

If there is some initial inventory, because of the concavity of the objective function, the optimal policy can be easily modified as follows: if the initial inventory is smaller than \( y^* \) it is optimal to order the difference, and if the initial inventory is larger than \( y^* \) it is optimal not to order at all. This result is the building block for the derivation of the base-stock policy in multi-period settings.

The optimum ordering quantity satisfies a balance equation of underage and overage costs. The nominator in (2.1), \( p - c \), represents the opportunity cost of losing one unit of demand because of ordering too few, so it is defined as the underage cost. Similarly, ordering one unit too much costs \( c \) which is the overage cost and the denominator is the sum of the two costs. Hence the optimum ordering quantity can also be represented as follows:

\[ y^* = F^{-1}\left(\frac{C_u}{C_u + C_o}\right) \tag{2.2} \]

where \( C_u \) and \( C_o \) are the underage and overage costs respectively. Even if the cost structure is different, by appropriately setting the overage and underage costs the optimum ordering quantity can be found using (2.2) given the basic properties of the model still hold. Two common extensions to the cost structure are including penalty cost and salvage value.

If each unit of leftover inventory can be sold out for a salvage value, the overage cost should be decreased by that value. If the salvage value is negative, that implies a disposal cost per unit of excess inventory. On the other hand, if there is per unit penalty cost associated with lost sales,
underage cost can be modified such that it includes the penalty cost. In both cases the solution in (2.2) can be used to find the ordering quantity.

The ratio of the overage and underage costs in (2.2) that determines $y^*$ is often called the critical fractile since it gives the fractile of demand distribution which corresponds to the optimum order quantity. Under the cost structure that we assume i.e. no salvage value and no penalty cost, the critical fractile can be written as:

$$Cr = \frac{p - c}{p}.$$

$Cr$ corresponds to the cycle-service level that the newsvendor aims to reach with the order quantity where cycle-service level is the probability of matching all the demand.

The critical fractile can also be interpreted as a measure for the profitability of the product. Schweitzer and Cachon (2000) classify products as high-profit products if $Cr > 1/2$ and as low-profit products otherwise. As $Cr$ increases the profit margin of the product increases and it becomes more valuable to sell one more unit. Thus, if the demand distribution remains the same, the expected profit maximizing order quantity $y^*$ increases as the profitability of the product increases. For example, if there are two products with the same cost structure and the same demand distribution, the order quantity of the one with the higher price should be larger.

### 2.2 Price-setting newsvendor model

In many applications, the newsvendor has the chance to price his own product at least within a specific range of prices. If this is the case, the most profitable way is to decide on the price and the ordering quantity simultaneously. The selling price $p$ becomes a decision variable and the demand is assumed to be dependent on price.

The need to consider pricing and inventory problems simultaneously was first pointed by Whitin (1955). He provides a newsvendor model with pricing assuming a stochastic price dependent demand function. He derives an optimality condition based on the relation between the expected marginal profit and the expected marginal cost. For a uniform demand distribution with a price dependent mean, he provides a closed-form expression for the optimal price, which is used to find the optimal order quantity.

The literature on the price-setting newsvendor model is mainly dominated by the demand definitions with an additive and/or multiplicative uncertainty.
In this section we will discuss these models in detail and the more general models will be discussed in chapter 4.

2.2.1 Modelling demand with additive and multiplicative uncertainty

Price dependency of demand in an environment with a stochastic demand process implies that at each price a different random variable might correspond to demand. Hence, the random demand is now represented as $X(p)$, and the distribution function as $F(p, x)$. The common practice is to represent $X(p)$ as a combination of a deterministic function and an error term. The admissible prices where $X(p)$ is defined satisfy $p_{\text{min}} \leq p \leq p_{\text{max}}$ where $p_{\text{min}}$ is generally assumed to be zero or $c$. $p_{\text{max}}$ is defined as the price at which expected demand is zero and it is possible that $p_{\text{max}} = \infty$. $d(p)$ is a nonnegative deterministic decreasing function of price, i.e. $d(p) \geq 0$, $d'(p) < 0$, and $U$ is a random variable with distribution function $\Phi(u)$ and density function $\phi(u)$ which are independent of price.

One of the most important issues about joint pricing and inventory models is the relation of $d(p)$ and $U$. Two typical approaches are to combine the two terms in an additive or a multiplicative fashion. The additive models correspond to the models where the demand is represented as the sum of the deterministic price dependent function and the random (error) term,

$$X^A(p) = d(p) + U. \quad (2.3)$$

On the other hand, the multiplicative models refer to the product of the two terms,

$$X^M(p) = d(p)U. \quad (2.4)$$

Under the additive model, probability density function of demand has the same shape as the density of the error term but its location is changed, while under the multiplicative model, the scale is changed. The distribution and the density functions of random demand can be written in terms of the corresponding functions of the error term. For the additive model $F(x^A, p) = \Phi(x^A - d(p))$ and $f(x^A, p) = \phi(x^A - d(p))$, and for the multiplicative model $F(x^M, p) = \Phi(x^M/d(p))$ and $f(x^M, p) = \phi(x^M/d(p))/d(p)$.

In the additive case, the mean value of the random term, $E[U]$, is generally assumed to be zero, and in the multiplicative case it is one. Thus, under both cases expected demand corresponds to the deterministic part, $E[X(p)] = d(p)$. It is common to assume $d(p) = a - bp$ with $a > 0$, $b > 0$ in the additive
models, and \( d(p) = ap^{-b} \) with \( a > 0, \ b > 1 \) in the multiplicative models (Petruzzi and Dada, 1999).

The main difference between additive and multiplicative models is the relation of price with the variance, \( \text{Var} \), and coefficient of variation, \( \text{Cv} \), of demand. Under the additive model, the demand variance is

\[
\text{Var}(X^A(p)) = \text{Var}(U)
\]

and \( \text{Cv} \) of demand is

\[
\text{Cv}(X^A(p)) = \frac{\sqrt{\text{Var}(U)}}{d(p) + E[U]}.
\]

Under the multiplicative model, the demand variance is

\[
\text{Var}(X^M(p)) = d(p)^2 \text{Var}(U)
\]

and the \( \text{Cv} \) is

\[
\text{Cv}(X^M(p)) = \frac{\sqrt{\text{Var}(U)}}{E[U]} = \text{Cv}(U).
\]

Hence, under the additive model the demand variance is constant in price while the coefficient of variation is increasing in price. On the other hand, under the multiplicative demand model, the coefficient of variation of demand equals that of the random term, which is independent of price, but the variance of demand is decreasing in price. This difference causes different pricing strategies depending on how the uncertainty is modelled. This point is the main focus of the earlier papers on the topic.

The combination of the two models is also considered in the literature in order to have a broader range of variability patterns. The demand is modelled as:

\[
X^C(p) = d_1(p)U + d_2(p)
\]  

(2.5)

If \( d_1(p) = 1 \), the formulation corresponds to the additive case, and if \( d_2(p) = 0 \) it corresponds to the multiplicative case. Under this model, the variance and the coefficient of variation of demand is

\[
\text{Var}(X^C(p)) = d_1(p)^2 \text{Var}(U)
\]

\[
\text{Cv}(X^C(p)) = \frac{d_1(p) \sqrt{\text{Var}(U)}}{d_1(p)E[U] + d_2(p)}.
\]
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Both \( Var \) and \( Cv \) depend on price, which provides a generalization to the previous two models. Young (1978) is the first paper that applies the combined demand model. However, in the succeeding works it is still preferred to consider the additive model and the multiplicative model separately with some exceptions like Chen and Simchi-Levi (2004b) in a multi-period setting.

When the uncertainty is modelled additively as \( X^A(p) = d(p) + U \), some conditions are required to guarantee nonnegative demand even if \( d(p) \geq 0 \) for all \( p \). For example, Yao et al. (2006) assume \( U \geq 0 \) even if they do not explicitly mention it and in order to avoid the infinitely large revenues, an upper bound is placed on the admissible prices, so if the revenues are increasing in price a finite optimal price is still ensured on the upper price bound.

On the other hand, Karlin and Carr (1962) does not bound the admissible prices or the error term and they mention that “Since it is tedious (though not difficult) to establish general conditions ensuring the nonnegativity of demand while at the same time retaining the assumption that \( U \) is distributed independently of the exogenous variable \( p \),...we shall be interested only in characterizing the properties of those solutions of the model for which the optimal policies involve positive price and ordering quantities. (We assume the existence of such solutions.)” Mills (1959), without mentioning this point, does not put any restrictions on the ranges of \( U \) or \( p \), so it seems like he also takes the approach of Karlin and Carr (1962).

Petruzzi and Dada (1999) use a specific demand function to study the additive model, \( d(p) = a - bp \) \( (a > 0, b > 0) \). In order to guarantee that the positive demand is possible for some prices, they assume a finite lower bound on the error term \( U \). However, while this bound ensures some positive demand, it does not guarantee the nonnegativity for all possible demand realizations in the range.

For the multiplicative model the random variable \( U \) is assumed to be nonnegative without any problems mentioned for the additive model. For the combined model Young (1978) assumes that the uncertainty is not additive for sufficiently large prices in order to avoid the nonnegativity problem. However, in this case it is not possible to say the combined model also covers the pure additive model.

It should be mentioned that this problem is not a crucial issue for practical applications since the price range is anyhow limited and it is not allowed to approach to infinity.
2.2.2 Maximizing the expected profit

When the price affects the demand process, the objective function of the newsvendor should be modified appropriately to include this effect.

Under the additive uncertainty model, the expected profit is written as

\[
\Pi^A(p, y) = \int_{-\infty}^{y-d(p)} p(d(p) + u)\phi(u)du + \int_{y-d(p)}^{\infty} p(y\phi(u)du - cy. \tag{2.6}
\]

As can be seen from the lower bound of the first integral, we do not assume a lower bound on \( U \) and write it as Karlin and Carr (1962) did. Of course, for a more precise expression the lower and, if necessary, the upper bound of the integrals can be different than (minus)infinity.

When the demand model has the multiplicative uncertainty, the expected profit is

\[
\Pi^M(p, y) = \int_{0}^{y/d(p)} p(d(p)u)\phi(u)du + \int_{y/d(p)}^{\infty} py\phi(u)du - cy. \tag{2.7}
\]

The objective is to find the optimal policy i.e. the optimal order quantity \( \tilde{y} \) and the optimal price \( \tilde{p} \), in order to maximize the expected profit in (2.6) or (2.7) depending on the demand model. This requires a joint optimization on two decision variables. The general approach is to solve the problem in a sequential way. First the quantity (price) is fixed, the price (quantity) is optimized and the resulting price \( p^*(y) \) (quantity \( y^*(p) \)) is plugged in the original expected profit function. The result is a univariate problem along the optimal price (quantity) path where the only decision variable is quantity (price). This function is then maximized with respect to quantity (price) to get the optimal quantity \( \tilde{y} \) for the joint maximization problem and the optimal price is then \( p^*(\tilde{y}) \).

In order to ensure the uniqueness and the existence of an optimal policy a series of assumptions are necessary both on the deterministic part of demand, \( d(p) \), and on the distribution of the random term, \( \Phi(u) \).

The first set of assumptions are necessary for the existence of a finite optimal price for a given inventory level. If \( p_{max} \) is finite even if the optimal price turns out to be a boundary solution it is still finite. On the other hand, if \( p_{max} \) is infinite the following should hold (see Karlin and Carr (1962), Young (1978)):

\[
\lim_{p\to\infty} d(p) = \lim_{p\to\infty} pd(p) = 0.
\]
For the uniqueness of an optimal price for a given inventory level, it is generally assumed that deterministic revenue is concave in price, i.e. $2d''(p) + pd'''(p) \leq 0$ (e.g. Young (1978), Zabel (1970)). However, this assumption implies that the models can not cover some commonly used demand functions like the isoelastic demand function $d(p) = ap^{-b}$. A slightly weaker assumption is that there exists a unique price which maximizes the deterministic profit, i.e. $(p - c)d'(p) + d(p)$ is continuous and has a unique positive zero (Karlin and Carr (1962), Petruzzi and Dada (1999)).

The uniqueness of the optimal price-quantity couple depends on the distribution of the random term. Young (1978) shows that if $U$ has a PF$_2$ distribution, i.e. $\phi(u)$ is log-concave, or if it has the log-normal distribution, the uniqueness property holds. Moreover the results can be extended to the combination of additive and multiplicative models. Petruzzi and Dada (1999) extends the result to the IFR distributions for the additive and the multiplicative models separately.

Yao et al. (2006) presents the most general assumptions for the multiplicative and the additive models. They employ two important concepts: the price elasticity of demand and the generalized failure rate (see chapter 4). They assume that the deterministic demand function has increasing price elasticity and the error term has strictly increasing generalized failure rate. Under these conditions both for the additive and the multiplicative models the optimal policy is unique.

2.2.3 Optimal price

The main focus of the papers on price-setting newsvendor problem is the structural properties of the optimal price. The inventory problem is generally considered as the result of the pricing decision and does not gain specific attention. As mentioned by Yano and Gilbert (2004), the earlier works take the deterministic demand functions as the starting point and then bring the uncertainty into question. That’s why they specifically focus on the effect of including uncertainty on pricing strategy.

Mills (1959) was the first to write the demand function explicitly as an additive demand model as in (2.3) and the main consideration is to show the effect of uncertainty on the optimal price. The price which optimizes the deterministic profit function is defined as the optimal riskless price $p^d$ such that

$$p^d = \arg\max_p \{(p - c)d(p)\}.$$
The optimal price which maximizes the expected profit with an additive demand function is

$$\tilde{p}^A = \arg\max_p \{\Pi^A\}.$$  

Mills shows that the optimal price under uncertainty is always smaller than the optimal riskless price, i.e. $$\tilde{p}^A < p^d.$$ Thus, introducing uncertainty in an additive way decreases the optimal price.

On the other hand, Karlin and Carr (1962) introduce the uncertainty in a multiplicative model and the resulting pricing strategy is opposite of the one shown by Mills (1959). Under multiplicative uncertainty the optimal price is higher than the riskless price, i.e. $$\tilde{p}^M > p^d,$$ where $$\tilde{p}^M = \arg\max_p \{\Pi^M\}.$$  

Young (1978) defined the demand function in a manner that combines both additive and multiplicative models, i.e. $$X^C(p) = d_1(p)U + d_2(p),$$ and verifies both results of Mills (1959) and Karlin and Carr (1962), and generalizes their results by describing the optimality conditions in terms of variance $$\text{Var},$$ and coefficient of variation $$\text{Cv}:$$

1. If $$\text{Cv}$$ of demand is non-increasing in price, optimal price is larger than the riskless price.

2. If $$\text{Var}$$ of demand is non-decreasing in price, optimal price is smaller than the riskless price.

However, when we look at the conditions it can be seen that they are still restricted in terms of variability pattern. If we define variability as the combined effect of the two measures, $$\text{Var}$$ and $$\text{Cv},$$ the above points represent the following properties:

1. If $$\text{Cv}$$ of demand is non-increasing, the $$\text{Var}$$ can only be decreasing in price. Both measures behave in the same direction, namely they are both non-increasing.

2. If $$\text{Var}$$ is non-decreasing, $$\text{Cv}$$ can only be increasing in price. Hence, both measures are non-decreasing in price.

For the multiplicative model, $$\text{Var}$$ is decreasing when $$\text{Cv}$$ is constant, and the first pattern is even a stronger decrease in variability. Likewise, the second pattern is the variability pattern of the additive model, and even stronger. Thus, the two conditions indicate the extreme cases and not a variability pattern where $$\text{Var}$$ and $$\text{Cv}$$ behave differently or where they are not monotone in price.

Petruzzi and Dada (1999) provide an intuitive explanation to the opposite behavior of optimal price under the additive and the multiplicative models.
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and its relation to variability. The main idea is that price is a measure to decrease the demand variability, but it works different under the two models. In the additive model, “it is possible to decrease the demand coefficient of variation without adversely affecting the demand variance by choosing a lower price”, on the other hand for the multiplicative model, “it is possible to decrease demand variance without adversely affecting the demand coefficient of variation by choosing a higher price”. As a result, it is intuitive that the optimal price should be lower in the additive model and higher in the multiplicative model than the deterministic price.

Throughout the analysis they use a transformation of the profit function by defining a safety factor $s$, and describe the optimal price as a function of $s$. For the additive case $s = y - d(p)$, and for the multiplicative case $s = y/d(p)$. In order to find a unifying condition for the additive and the multiplicative models they write the profit function as

$$\bar{\Pi}(s, p) = (p - c)E[Sales(s, p)] - cE[Leftovers(s, p)].$$

If $p_B(s)$ is the base price that maximizes the first part of the function, i.e. $p_B(s) = \arg\max_p (p - c)E[Sales(s, p)]$, the relation between the base price $p_B(s)$ and the optimal price $\bar{p}$ is the same under both types of uncertainty. They show that for the additive demand model

$$p^d \geq \bar{p}^A = p_B(s)$$

and for the multiplicative demand model

$$\bar{p}^M \geq p^d = p_B(s).$$

Thus even if the relation to the deterministic price is different under the additive and the multiplicative uncertainty, the optimal price is larger than or equal to the base price for both types of demand uncertainty.

The problem of defining the lower bound for error distribution and its consequence on the optimal price is mentioned by Van Mieghem and Dada (1999). For a specific additive demand definition $X(p) = -p + u$, they show that the optimal price can be higher or lower than the riskless price which is contradicting with the rest of the literature. They do not assume a lower bound on $u$ but they partition the state space for $u$ such that in one of the domains $u < p$ and hence demand is negative. They then assume in this domain $X(p) = 0$. However, since the range of this domain depends on price the distribution of $u$ becomes dependent on price. This means, on the other hand, that the uncertainty is not additive and the result can not be
compared to the results under additive models. This example shows the
problems with defining a pure additive demand model.

While the effect of uncertainty on price is discussed commonly in the
literature, there are fewer results about the effect on order quantity. The
effect depends also on the cost parameters and the properties of the error
distribution. Hence unifying results are not available.