Chapter 3

An Empirical Study

The review of the price-setting newsvendor model shows that the analytical results and the structural properties of the model depend on the type of the price-variability relation. Especially the different variability patterns underlying the additive and the multiplicative models influence the pricing decision in opposite directions. While the two models present nice technical properties and ease of use, when we turn from modelling to application it becomes critical to find the most appropriate model for the underlying real demand process.

Within the inventory control literature the relevant properties of the demand process are generally assumed to be known. In case of the price-setting newsvendor model, the parameters of the demand function and the probability distribution are considered as given and the estimation process is not treated explicitly. On the other hand, the use of additive and multiplicative models is motivated by their popularity in the marketing literature and in practice for the demand estimation process.

Especially in the marketing literature when the effect of price on the total amount of demand is considered, the most commonly used formulations are the additive and the multiplicative models. A linear regression of demand on price represents the additive model, and the linear regression of the logarithm of demand on the logarithm of price implies the multiplicative model. We will refer to the demand models which are used to represent the underlying unknown demand process as the additive and the multiplicative form while the regression models built to estimate them will be called the linear form and the log-linear form, respectively. The two forms of regression models are frequently employed for estimating the price-demand relation. In a meta analysis of econometric studies between 1960 and 1985 about price elasticity of sales Tellis (1988) reports that the linear and the log-linear forms are among the most often utilized forms. Extending the work of Tellis (1988) with studies up to 1991, the same observation is revealed by Kalyanam (1996). However, it is important not to forget the aim of these studies and the problem for which these models are used.

The relevant models from the marketing literature are developed in order to estimate price sensitivity of demand and to come up with a pricing policy.
The optimal prices are calculated based on expected sales which is assumed to be the same as expected demand. This implies two major assumptions; 1) the variability regarding the impact of price on demand can be captured by the functional form selection 2) the inventory level is not considered or assumed to be unlimited for the calculation of expected sales.

If the true demand process follows a multiplicative or an additive form, with a log-linear or a linear regression, the effect of price on variance can be correctly captured. Without knowing the exact structure of the underlying process the selection of the functional form for the regression model becomes critical. However, defining the selection criteria and the tests or the methods is not trivial. One way of overcoming this problem is to use more flexible methods for estimation. For example, Kalyanam (1996) suggests the use of a Bayesian mixture model where the different functional forms are considered at the same time with some appropriate probabilities and the optimal prices are calculated based on this mixture model. However, the goal is still maximizing expected revenues assuming that the expected sales is equal to the expected demand.

In such settings, modelling or forecasting mean demand is the main question, where the focus is on creating good point forecasts while the effect of variance has a secondary role. This is justifiable as long as the profits or revenues are determined just by mean demand and stockouts are not considered. The effect of variability can be then safely ignored since positive and negative deviations from mean demand can compensate in the long run. When we start considering a limited available quantity to offer to the market, positive and negative demand deviations can not compensate anymore because of stockout situations. Hence, variability of demand starts having a major impact and we need to consider expected sales instead of expected demand for optimization models.

Nevertheless, one of the motivations for using the additive and the multiplicative models in the price-setting newsvendor problem is the common usage of these models by practitioners and the claim that the two models can represent demand in an appropriate way in many cases. In this chapter we question this claim and investigate if one of the models represents the demand process of different products better than the other.

For this purpose, we present an empirical study based on sales data from a retail chain company for several consumer products. First, we check if the properties of the additive and the multiplicative models, especially in terms of price-variance relation, hold for the data at hand. In order to estimate the demand functions we use regression analysis which is the most commonly employed statistical tool for research on pricing (Brown and Dant, 2008).
Second, and more important, we analyze the effect of the different models on the resulting profit. We apply the newsvendor model for the ordering and pricing decisions by using both demand models, compare the resulting policies, and check if the better-fit model (let us define “better-fit model” as the model which represents the data better in terms of some statistical measures) leads to significantly higher profits.

It should be underlined that the main interest is the effect of these models on the expected and real profits. Fitting of the two models to the empirical data will be, of course, evaluated with statistical measures and will give the basic comparison criterion. However, if the profit implications of using the better or worse fitting model is not significant, the contradicting analytical results of the two models with respect to the pricing policy can be considered loosely in practical settings.

As a third point we suggest a more flexible form which can capture different forms of variability. The pricing and inventory policy and the profits are calculated and compared against the two classical forms.

3.1 Description of the data

For our analysis we use weekly sales data of an Austrian retail chain company from a number of outlets for several products on the stock-keeping unit (SKU) level. We started with 11941 products which are sold at least for 52 weeks and on average 5 units per week. The sales data do not correspond to the real demand since demand during stockouts is not recorded. However, an analysis of stockout situations on the real data shows that they occur in less than 2% of the selling periods. Therefore, the existing sales data can be considered as an indicator for demand.

At each period, the products can be sold with different prices in different outlets. However, our analysis is not on outlet level but on the aggregated sales of each product. In order to come up with a single price at a single period, a weighted average price is calculated: the price charged at each outlet is weighted by the proportion of demand in that outlet to the overall demand. For further analysis, prices are fitted to a grid of 10 equally distributed steps.

Next to sales price, the data also contain information about the number of market outlets, and a features indicator (binary information to account for the effect of advertisement, e.g. by means of newspaper supplements such as flyers and leaflets). Just for a quarter of products such advertisement efforts were utilized. As we do with price, we also take a weighted average of the
binary feature indicator over all outlets. Therefore at the end the variable that we use for feature is no more a binary variable but continuous between 0 and 1.

Since we are interested in the variability pattern for different prices, we want to be able to calculate the variance of demand for a number of different price levels. Therefore we eliminated the products where there are less than five prices with at least five observations each. For a given price level if there are at least five observations we consider that price for calculating the demand variance, otherwise there are not enough observations to calculate the variance. On the other hand if there are less than five prices with enough observations, there are too few prices where we can calculate the variance and analyze the effect of price on variance. Because of this reason a total of 7462 products are eliminated, and finally we proceed with 4479 products for the subsequent analysis.

Note that the products of the company are in fact durable goods, so the inventory control policy can be better modelled with a multi-period problem formulation. However we use the data to test optimal pricing and inventory policies within the newsvendor framework, without comparing our results with the applied policy at the retailer.

3.2 Demand estimation

As we are interested in the price-demand relation we want to model demand only as a function of price. In reality, however, demand depends on a number of other factors which should be identified. By cleaning the data from these factors we can derive the price dependent part of demand. In the following subsections we describe the estimation procedure for a single product, while the same procedure is applied to each product.

3.2.1 Detrending demand data

The potential factors that can significantly affect demand are chosen as trend, seasonality, number of outlets and features (see Natter et al. (2007) for a discussion). As the first step, the random demand for the product is formulated as a function of these variables:

\[ D = z^T \beta + X \]  

(3.1)
3.2 Demand estimation

where \( z \) is a column vector of normalized independent variables such that the average over all observations of each variable is zero, and \( z^T \) is its transpose,

\[
    z^T = \left( t, \sin \left( \frac{2t\pi}{52} \right), \cos \left( \frac{2t\pi}{52} \right), \sin \left( \frac{2t\pi}{43.3} \right), \cos \left( \frac{2t\pi}{43.3} \right), O, F \right). \tag{3.2}
\]

The first term \( t \) captures the time trend and the next four terms are the seasonality components, with both annual and monthly cycles, \( O \) is the number of outlets and \( F \) is the feature variable. The column vector \( \beta \) is the coefficient vector with parameters \( \beta_1 \ldots \beta_7 \). As it can be seen from (3.2), the regressors (the independent variables to the regression model) do not include the constant and the price, so the effects of these two are still captured in \( X \). Hence, \( X \) corresponds to the price dependent demand, i.e. \( X(p) \).

Let \( \hat{\beta} \) be the vector of sample estimates of \( \beta_i \) for \( i = 1, \ldots, 7 \) and let \( \hat{x}_k \) be the residuals to estimate \( x \). By using the least squares estimation method we solve

\[
    D_k = z_k^T \hat{\beta} + x_k. \tag{3.3}
\]

We use \( k \) as an index for a single observation, where total number of observations is \( N \). The optimal values for \( \hat{\beta}_i \) are found in order to minimize \( \sum_{k=1}^{N} (x_k)^2 \), and they are checked for significance on a 5% level. Note that not all the regressors are used to estimate the final model. Starting from the most general model, i.e. including all the elements of \( z \), we eliminate the insignificant variables step by step. While there are a number of different approaches to model building, we find the general-to-simple approach good enough for our purposes. The estimates of the coefficients of the insignificant variables are set to 0 in \( \hat{\beta} \).

Now \( x_k, k = 1, \ldots, N, \) is the data cleaned from any effect other than price, which we need for our further analysis. Using these data we fit an additive and a multiplicative demand model again using the linear regression models and the least squares estimation method.

3.2.2 Estimating the additive and the multiplicative models

From this point on, we fit the prices to a grid of 10 equally distributed steps. Let \( p_{\text{min}} \) and \( p_{\text{max}} \) be the minimum and the maximum observed prices. The price grid \( \rho \) has the elements \( \rho_1, \ldots, \rho_{10} \) such that \( \rho_i - \rho_{i-1} \) is the same for all \( i \) where \( \rho_1 = p_{\text{min}} \) and \( \rho_{10} = p_{\text{max}} \). The price for \( k^{\text{th}} \) observation, \( p_k \), is set to the closest price in the grid.
The additive model is formulated as,

\[ X(p) = a + bp + U^A. \]

In order to estimate the coefficients \(a\) and \(b\) the following regression model is solved for the respective estimates \(\hat{a}\) and \(\hat{b}\).

\[ x_k = \hat{a} + \hat{b}p_k + \hat{u}_k. \quad (3.4) \]

The multiplicative model is formulated as,

\[ X(p) = mp^nU^M \]

and the estimates are obtained through

\[ \ln(x_k) = \ln(\hat{m}) + \hat{n}\ln(p_k) + \hat{u}_k. \quad (3.5) \]

Note that in (3.5) \(\hat{u}_k\) is an estimate for \(\ln(u^M)\).

After the estimation of the regression coefficients, they are checked to ensure that they fit with the basic theoretical assumptions. First of all, if for a given product, demand turns out to be increasing in price we do not consider that product for further analysis. While there can be some class of products (such as luxury goods) where this relation fits with the theory, given the product portfolio of the retailer that we consider, it is quite unlikely to come up with such products. Moreover the pricing strategy of such products are not covered yet by the price-setting newsvendor literature and therefore is out of our scope.

For the linear model, if \(\hat{b}\) turns out to be positive demand is increasing in price. Therefore, the linear form is not considered for comparison and in the optimization problem for all those products where \(\hat{b}\) is not significantly smaller than zero. There are 3223 products, or 72\% of all products where the regression analysis was applied, which results in a linear demand function decreasing in price.

For the log-linear model, if \(\hat{n}\) turns out to be positive, demand is increasing in price. Moreover, if \(-1 \leq \hat{n} \leq 0\) demand is inelastic to price changes. As a result, the model results in a profit which is monotonically increasing in price. When the only intention is making point estimates of demand or deriving properties of demand elasticities, \(\hat{n}\) does not need to be elastic. However, when the problem is optimizing prices using the estimates, elasticity is required in order to be able to get finite optimal prices. This problem with the log-linear model is also mentioned and further discussed by Montgomery.
and Bradlow (1999). Hence, if $\hat{\alpha} \geq -1$, the log-linear form is not considered for further analysis. Quite a high number of products, namely 2136 products, fall in this case. This corresponds to 48% of the products which are considered for the regression analysis.

These criteria about the two models can be considered as the face validity conditions, since the results are compared with theoretical expectations. A total of 1232 products fail to satisfy any of these conditions and, therefore, these products are not considered for further analysis. On the other hand, 2319 products satisfy the conditions for both of the models. The number and percentage of products satisfying the conditions for the two models are given in Table 3.1.

One of the main problems with time series data is the autocorrelation of the residuals across periods. A common cause of autocorrelation is model misspecification (Verbeek, 2008, p.105). In marketing, autocorrelation from model misspecification is generally the result of omission of a relevant explanatory variable (Hanssens et al., 2003, p.215). For example, the prices of complementary and substitutable products might have a significant effect on demand. However, since we do not know about the interrelations between products we are not able to model these effects, which might cause the misspecification problem.

In order to check for autocorrelation we apply the Durbin-Watson test which is a popular test for first-order autocorrelation (Verbeek, 2008). For the model in (3.4) the test statistic is:

$$ dw = \frac{\sum_{k=2}^{N} (\hat{u}_k - \hat{u}_{k-1})^2}{\sum_{k=1}^{N} \hat{u}_k^2}. $$

A non-autocorrelated process results in a test statistic $dw = 2$. Hence, we test for $dw$ being significantly different from 2. We do not check for the higher-order autocorrelation since generally the amount of autocorrelation diminishes with the lag (Ledolter and Abraham, 2006), and the seasonality

### Table 3.1: Number of products which satisfy the face validity condition

<table>
<thead>
<tr>
<th>Model</th>
<th>Num. products</th>
<th>% of prod.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Additive</td>
<td>3223</td>
<td>71.9</td>
</tr>
<tr>
<td>Multiplicative</td>
<td>2343</td>
<td>52.3</td>
</tr>
<tr>
<td>Both</td>
<td>2319</td>
<td>51.7</td>
</tr>
<tr>
<td>None</td>
<td>1232</td>
<td>27.5</td>
</tr>
</tbody>
</table>
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is cleared in the detrending step which can help reduce the possible seasonal autocorrelation.

The results of the Durbin-Watson test for each model is as follows: out of the products satisfying the face validity for the additive model, 1092 (33.9%) are non-autocorrelated. For the multiplicative model, 623 (26.6%) products satisfying the face validity are non-autocorrelated. 550 (23.7%) products satisfy the face validity of both models and at the same time are non-autocorrelated. For the rest of the chapter, we first report the results for all the products that pass the face validity check followed by the results for the non-autocorrelated products (which will be denoted as “nae.”).

For all products that passed the face validity check, the histogram of the $R^2$ measures are given in Figure 3.1. While it seems like the $R^2$ values for the log-linear model are generally higher compared to the linear one, we should take into consideration that the linear model is estimated for a larger number of products compared to the log-linear model. There are 852 products with $R^2 \geq 0.2$ for the linear model, while it is 693 for the log-linear. These numbers correspond to 27% of the products for which the linear model is estimated and 29% for the log-linear model. When we consider just the nae. products, the picture is not very different. Figure 3.2 shows the histogram for the $R^2$ values just for the non-autocorrelated products.

The $R^2$ values seem to be quite low for both models, but the reason is not the inadequate fit of the estimates to the real demand data but the fact that we are just looking at the price dependent part. Let $R^2_{comp}$ denote the

Figure 3.1: Histogram of $R^2$ of the linear (left) and the log-linear (right) models
3.3 Selection among the additive and the multiplicative models

When both the linear and the log-linear forms are candidates for the demand model an important question is which one to select since the two forms will obviously suggest different pricing and inventory policies. Hence the models should be compared against each other in order to identify the most representative model. While for a small number of products it is possible to manually look at many aspects in detail, once we have a realistic assortment

\[
R^2_{\text{comp}} = 1 - \frac{\sum_{k=1}^{N} \hat{u}_k}{\sum_{k=1}^{N} (D_k - \bar{D})^2}.
\]

The histogram of \(R^2_{\text{comp}}\) is given in Figure 3.3 which shows an obvious improvement over the \(R^2\) values in Figure 3.1 and Figure 3.2.

Figure 3.2: Histogram of \(R^2\) of the linear (left) and the log-linear (right) models for nac. products

\(R^2\) when we take the complete estimate and compare it with the original demand. If we consider the \(R^2_{\text{comp}}\) values we see that they are indeed quite reasonable.

For the additive model we calculate the complete fit of the estimates as

\[
R^2_{\text{comp}} = 1 - \frac{\sum_{k=1}^{N} \hat{u}_k}{\sum_{k=1}^{N} (D_k - \bar{D})^2}.
\]
size with thousands of products we need to select the criteria to base our decision on.

Since the dependent variables of the two models are not the same, it is not appropriate to use $R^2$ directly or the likelihood function values as criteria for comparison. One possible way of comparison is transforming the estimate of the log-linear model back to the original demand level and calculating the $R^2$ based on this estimate. However, the transformation is not very straightforward. One has to consider the distribution of the error term and the distribution on the estimated coefficients. Thus the result is sensitive to the distributional assumptions and there is no clear convenient way of doing a comparison on $R^2$.

First we employ a standard procedure for comparing linear and log-linear models using a general purpose test for model selection. Next, since a critical issue in our work is the price-variability relation, we base the model comparison on how good the demand variance is captured by each functional form.

### 3.3.1 A formal test for model selection

One of the two typical approaches for comparing linear and log-linear forms is nesting them under a more general Box-Cox transformation and comparing them against this general model (see e.g. Davidson and MacKinnon (2004,
3.3 Selection among the additive and the multiplicative models

The second approach is to create an artificial regression model using the two models such that both are nested under this artificial model. In this section we take the latter approach and use the $P_E$-test suggested by MacKinnon et al. (1983) for the comparison.

The $P_E$-test procedure is based on estimating an artificial compound model from the two competing models and testing the null hypothesis $H_0$ which states that the correct model is the linear model

$$H_0 : x_k = a + bp_k + u_k.$$ 

against the alternative $H_1$ which says that the correct model is the log-linear model

$$H_1 : \ln(x_k) = \ln(m) + n \ln(p_k) + \nu_k.$$ 

Based on the two models the artificial compound model is generated as

$$x_k = (a + bp_k) + a\{(\ln(m) + \hat{n} \ln(p_k)) - \ln(\hat{a} + \hat{b}p_k)\} + \zeta$$

The parameters $\hat{a}$ and $\hat{b}$ are the estimates from the linear model in equation (3.4) and $\hat{m}$ and $\hat{n}$ are the estimates from the log-linear model in equation (3.5). The idea is estimating $\alpha$ and reaching a conclusion based on this estimate with a standard t-test. If $\alpha$ is significantly different from zero, the linear model is rejected, but this does not mean that the log-linear model has to be accepted. As an opposite case, if $\alpha$ does not turn out to be significantly different from zero, then the linear model is not rejected, but again, we can not yet conclude that the log-linear model is not also true. Up to this point the test is not very conclusive, hence the same procedure is repeated after exchanging the null and the alternative hypothesis. Now $H_0$ corresponds to the log-linear model and $H_1$ to the linear model and the artificial compound model is also reformulated accordingly as

$$\ln(x_k) = (\ln(m) + n \ln(p_k)) + \alpha\{(\hat{a} + \hat{b}p_k) - (\hat{m}p_k)\} + \zeta$$

At the end the test might give one of the four possible results as shown in Table 3.2. As can be seen, the problem with the $P_E$-test is that in two cases (type 3 and 4) the result is not conclusive. The test is not able to really compare the two models against each other but "the goal is to assess the 'truth' of each model's specification and categorically accept or reject each of the competing alternatives" (Balasubramanian and Jain, 1994).

Table 3.2 includes the number of products which fall into each category for all products and for the non-autocorrelated products, respectively. Un-
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Table 3.2: Results from $P_E$-test

<table>
<thead>
<tr>
<th>Type</th>
<th>Conclusion drawn</th>
<th>Nr. and % of products</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Accept linear, reject log-linear</td>
<td>281 (12.1%)</td>
</tr>
<tr>
<td>2</td>
<td>Reject linear, accept log-linear</td>
<td>282 (12.2%)</td>
</tr>
<tr>
<td>3</td>
<td>Reject both linear and log-linear</td>
<td>598 (25.8%)</td>
</tr>
<tr>
<td>4</td>
<td>Reject neither linear nor log-linear</td>
<td>1158 (49.9%)</td>
</tr>
</tbody>
</table>

Fortunately many of the products are of type 3 or 4, which are not very conclusive. Moreover, the number of products of type 1 and 2 are very close. Hence, the conclusion from this test can only be that a priori neither of the models is more preferable against the other one.

Similar results are reported by Bolton (1989) and Kalyanam (1996). Bolton (1989) compares the models based on the transformed $R^2$ values while he does not comment on how the transformation was done. He concludes that the average $R^2$ is approximately equal across models. However, the different functional forms have different systematic bias with respect to the estimates. Therefore it is crucial to test the alternative forms. On the other hand Kalyanam (1996) compares the log-linear model with a semi-log model, i.e. a model in the form of $\ln(x_k) = \beta_0 + \beta_1 p + \epsilon$, by using the P-test which is based on an artificial compound model like the $P_E$-test. He faces the same problem of the low power of the test since for 5 out of 6 comparisons the result is of type 4, so no conclusion can be made.

3.3.2 Selection based on homoskedasticity

While the $P_E$-test is a formal way of comparing the linear and the log-linear models in order to find the true model, evaluating them based on a specific criterion can be another legitimate way of comparison. In this section, we test how good the models represent the demand variance and use this as a criterion for model selection.

Recall that within the price-setting newsvendor model, the main effect of using different demand models come from the different variability patterns implied by these models. Under the additive model demand variance is assumed to be independent of price, and under the multiplicative one the coefficient of variation should be price independent. In order to investigate these two properties, we analyze the variance of the residuals of the two regression models in (3.4) and (3.5). If the corresponding residuals, $\hat{u}_k$ and/or
3.3 Selection among the additive and the multiplicative models

Table 3.3: Number and percentage of heteroskedastic products

<table>
<thead>
<tr>
<th>Model</th>
<th>All products</th>
<th></th>
<th>Nac. products</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>B-P test</td>
<td>White test</td>
<td>B-P test</td>
<td>White test</td>
</tr>
<tr>
<td>linear</td>
<td>1566 (48.6%)</td>
<td>1435 (44.5%)</td>
<td>443 (40.6%)</td>
<td>384 (35.2%)</td>
</tr>
<tr>
<td>log-linear</td>
<td>546 (23.3%)</td>
<td>635 (27.1%)</td>
<td>93 (14.9%)</td>
<td>100 (16.1%)</td>
</tr>
</tbody>
</table>

\( \hat{\nu}_k \), are homoskedastic, i.e. the variance of the residuals is identical through the observations, the assumptions of the additive and/or the multiplicative models are fulfilled. We check for homoskedasticity of the residuals by using some standard tests, namely the Breusch-Pagan test (B-P test) and the White test. Both of these tests aim to identify the heteroskedasticity which might be caused by one of the independent variables.

The test by Breusch and Pagan (1979) is based on an auxiliary regression of the squared residuals on a function of price. If we consider the additive case, the auxiliary regression is:

\[ \sigma^2_k = \sigma^2 f(\alpha_0 + \alpha p_k), \]

where \( \sigma^2 \) is a constant independent of the observation and \( f \) is any general function. If \( \alpha = 0 \), \( \sigma^2_k \) is the same for all observations and the model is homoskedastic. Hence, the null and alternative hypotheses are:

\[ H_0 : \alpha = 0 \]
\[ H_1 : \text{Not } H_0. \]

The test requires the specification of \( f \) and for this purpose we choose the simplest variant, where \( f \) is a linear function in \( p \). Since the Breusch-Pagan test is quite sensitive to the assumption of normality, we use the extension introduced by Koenker (1981) and Koenker and Bassett (1982).

The test by White (1980) is a generalization of the B-P test such that \( f \) is a linear function of \( p \) and \( p^2 \). Hence, it can cover non-monotone variance changes in price. While this generalization offers an advantage, it results in a limited power of the test.

The results of the two tests are reported in Table 3.3. Note that the tests were applied to the products which satisfy face validity, so the number of products under the linear and the log-linear models differ (see Table 3.1). When we look at the percentages, it can be concluded that the log-linear model seems to capture the price-variance relation better than the linear
Table 3.4: Comparison based on tests of heteroskedasticity: number and percentage of products in each type

<table>
<thead>
<tr>
<th>Type</th>
<th>Linear</th>
<th>Log-linear</th>
<th>B-P test (%)</th>
<th>White test (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Homosked.</td>
<td>Heterosked.</td>
<td>271 (18.0%)</td>
<td>248 (18.1%)</td>
</tr>
<tr>
<td>2</td>
<td>Heterosked.</td>
<td>Homosked.</td>
<td>964 (64.1%)</td>
<td>740 (54.1%)</td>
</tr>
<tr>
<td>3</td>
<td>Heterosked.</td>
<td>Heterosked.</td>
<td>268 (17.8%)</td>
<td>381 (27.8%)</td>
</tr>
<tr>
<td>4</td>
<td>Homosked.</td>
<td>Homosked.</td>
<td>817</td>
<td>950</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Type</th>
<th>Linear</th>
<th>Log-linear</th>
<th>B-P test (%)</th>
<th>White test (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Homosked.</td>
<td>Heterosked.</td>
<td>46 (16.5%)</td>
<td>39 (16.1%)</td>
</tr>
<tr>
<td>2</td>
<td>Heterosked.</td>
<td>Homosked.</td>
<td>197 (70.9%)</td>
<td>160 (66.1%)</td>
</tr>
<tr>
<td>3</td>
<td>Heterosked.</td>
<td>Heterosked.</td>
<td>35 (12.6%)</td>
<td>43 (17.8%)</td>
</tr>
<tr>
<td>4</td>
<td>Homosked.</td>
<td>Homosked.</td>
<td>272</td>
<td>308</td>
</tr>
</tbody>
</table>

1. Percentage values are calculated based on the sum of products of type 1, 2, and 3.

For the products which satisfy the face validity for both models, we can have a more detailed look in Table 3.4. While the tests are not comparing the two models, we use the results to interpret them for a comparison with respect to the truth of the model as we did with the $P_E$-test in the previous section. In Table 3.4 type 1 and 2 recommend using the linear or the log-linear model, respectively, if the only criterion would be about variance. Type 3 on the other hand includes all those products where neither the linear nor the log-linear model is able to capture the full price-variance relation. Type 4 products include all those where both the linear and the log-linear model result in homoskedastic residuals. Although this last category does not sound to be very intuitive, one of the following points can be the explanation:

1. The demand and its variance does not significantly depend on price.
2. Because of the low power of the tests the null hypothesis of homoskedasticity is not rejected.
3. Due to the range of the observed prices and demands, the log-linear model might look very similar to the linear one and hence, the test is not able to differentiate between them.
3.3 Selection among the additive and the multiplicative models

4. The variance in fact depends on price in both cases but can not be captured with a simple linear relation as in the B-P test or with a second order polynomial as in the White test. Since the type 4 result is quite vague and inconclusive, we do not include it in the calculation of the percentages. When we consider the first three types we can conclude that the log-linear model is again performing better than the linear one, which is in line with many findings.

The results are slightly different for the non-autocorrelated products as the proportion of type 3 products is smaller while the type 2 is higher for these products. Nevertheless the conclusion is not different than the conclusion considering all of the products: the log-linear model performs better than the linear but for many products it is not possible to conclude for one of the models.

3.3.3 Summary of model selection

The comparisons based on both the PE-test and homoskedasticity result in a large class of products where it is not possible to choose among the linear and the log-linear models. First of all, there are a number of products which fall in type 3 both in Table 3.2 and Table 3.3 for which neither of the models seem to be appropriate. In Figure 3.4 we plot the variance of demand for two of these products. The first product does clearly not fit with the assumptions of neither of the models since it has an increasing variance,
while the second one seems to imply a non-monotone price-variance relation. Therefore it is no surprise that they are identified as heteroskedastic under both the linear and log-linear forms.

Another problematic class of products is that one, where the log-linear model is identified as homoskedastic (i.e. type 2 in Table 3.3) because the variance is decreasing in price, while the coefficient of variation in fact does not fit with the multiplicative model. Figure 3.5 shows the Var and Cv of one of these products. Since $\text{Var}(X(p))$ is decreasing in $p$, both the B-P test and the White test identified it as homoskedastic under the log-linear model. Hence, one can assume that the findings from the multiplicative model would apply for this product. However, this might be a misleading conclusion because $\text{Cv}(X(p))$ is not really constant in $p$ as implied by the multiplicative model.

3.4 Fitting a general model

As the previous discussion and analysis remained unsatisfactory with respect to several aspects, we see the need for a more general and flexible demand definition. There are numerous products where the adequacy of the linear and the log-linear forms are questionable, so we try to model demand in a way that we can cover the properties of demand for products as in Figure 3.4 and 3.5. Since our aim is to find the optimal price and inventory level with the newsvendor model we do not need a point estimate but a whole distribution.
3.4 Fitting a general model

Therefore, already at the estimation step focusing on the distribution seems to be reasonable. For now, we leave the discussion of which distribution to choose and focus on the specification of the two parameters the mean and the variance of the distribution.

We suggest estimating the variance and the mean of demand for each price separately, and using directly the distribution function with these parameters. Since we want to be able to capture a more general price-variance relation we model \( \text{Var}(X(p)) \) as:

\[
\text{Var}(X(p)) = \beta^0 + \beta^1 p + \beta^2 p^2. \tag{3.6}
\]

With this variance function, we can capture any monotone price-variance relation as well as non-monotone relations with one maximum or minimum point. In order to estimate the coefficients in equation (3.6) we use the sample variance \( \text{Var}(X(\rho_i)) \). First we need to calculate the variance of sample demand for each \( \rho_i \in \mathbf{p} \), but for some \( \rho_i \) we do not have enough number of observations to come up with the variance. Hence, we calculate the sample variance for \( \rho_i \) if there are at least 5 observations with that price, and if there are less than 5 observations the variance is found by a linear interpolation of the available variances. Remember from Section 3.1 that we just consider products with at least 5 prices where we can calculate the variance. In this way we come up with \( \text{Var}(X(\rho_i)) \) for all \( i = 1, \ldots, 10 \). Then, we write \( \text{Var}(X(\rho_i)) \) as a price dependent function:

\[
\text{Var}(X(\rho_i)) = \beta^0 + \beta^1 \rho_i + \beta^2 \rho_i^2 + \omega_i. \tag{3.7}
\]

Again with least squares method we estimate the coefficients of the regression equation (3.7). When we apply this procedure the resulting variance functions for products in Figure 3.4 are illustrated in Figure 3.6.

We do not estimate a specific function in order to find out the mean demand, and we directly use the average sample demand for each \( \rho_i \) where there are at least 5 observations and the others are linearly interpolated. At the end, we come up with a variance, \( \text{Var}(X(\rho_i)) \) and mean \( E(X(\rho_i)) \) for each price \( \rho_i \). We will use this model as a benchmark for evaluating the potential profit improvements of using a model which is more general than the additive and the multiplicative models.

Up to this point we modelled demand in three different ways and estimated the parameters of these models for each product. We evaluated the additive and the multiplicative models using statistical tests focusing on the correct representation of the variance. In the rest of this chapter we apply the
price-setting newsvendor model for each product using the three demand models. We evaluate and compare the optimal policies resulting from each model, and using the empirical observations we simulate the policies to calculate and compare what we call the real average profits.

### 3.5 Simulation of profits

In this section we find the optimal price and inventory levels under different demand models using the newsvendor formula. Thus we need to calculate the expected profits which require specification of the cost parameters. Since we do not have access to the information about the real cost structure, we made some assumptions: 1) the unit purchasing cost is set as 80% of the smallest observed price, i.e. $c = 0.80p_1$, 2) there is no salvage value for leftover inventory and no penalty cost for unsatisfied demand.

In order to calculate expected profits we need the distribution function of demand. Hence, we should specify a price dependent distribution function $F(p, x)$ corresponding to each of the three demand models.

For the additive and the multiplicative models we need the distribution of the error terms $U^A$ and $U^M$ (see equations 2.3 and 2.4) to derive the demand distribution. The first step is to estimate the distribution of the residuals $\hat{u}$ and $\hat{v}$ from the linear and the log-linear models. While it is theoretically possible to find the best fitting distribution separately for each

![Figure 3.6: Sample variance (dashed) and fitted variance (solid) of two example products](image)

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3.5 Simulation of profits

Table 3.5: Summary of the demand distributions under the three models

<table>
<thead>
<tr>
<th>Model</th>
<th>Distribution</th>
<th>Mean</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Additive</td>
<td>Normal</td>
<td>$\hat{a} + \hat{b}p$</td>
<td>$\text{Var}(U^A)$</td>
</tr>
<tr>
<td>Multiplicative</td>
<td>Log-normal</td>
<td>$\hat{m}_p \hat{n} E(U^M)$</td>
<td>$\hat{m}_p \hat{n} \text{Var}(U^M)$</td>
</tr>
<tr>
<td>General</td>
<td>Normal</td>
<td>Sample mean, $E(X(\rho))$</td>
<td>$\beta^0 + \beta^1 p + \beta^2 p^2$</td>
</tr>
</tbody>
</table>

product and for each functional form, we assume a normal distribution both for $\hat{u}$ and $\hat{v}$ for all the products.

The distribution of $U^A$ is directly estimated as a normal distribution with the mean and the variance of $\hat{u}$:

$$E(U^A) = 0 \quad \text{Var}(U^A) = \text{Var}(\hat{u})$$

The distribution of $U^M$ is estimated as a log-normal distribution with the mean and variance of $\hat{v}$:

$$E(U^M) = e^{\text{Var}(\hat{v})/2}$$
$$\text{Var}(U^M) = e^{\text{Var}(\hat{v})}(e^{\text{Var}(\hat{v})} - 1).$$

Using these parameters the demand distribution under the additive and the multiplicative models are derived. The corresponding distribution functions are denoted as $F^A(p, x)$ and $F^M(p, x)$ respectively. For the general model we assume a normal distribution with the sample mean and the variance as in Equation (3.6) and the distribution function $F^G(p, x)$. Table 3.5 summarizes this part showing which distribution is used under each model and lists the corresponding mean and variance.

After specifying the distribution functions and the parameters, first we find the optimal inventory level for each price in the observed range, i.e. for all $\rho_i$, using the explicit formula in equation (2.1). For example, for the additive model optimal inventory level satisfies

$$F^A(\rho_i, y^A^*(\rho_i)) = \frac{\rho_i - c}{\rho_i}$$

and the expected profit using the $y^A^*(\rho_i)$ is

$$\Pi^A(\rho_i) = (\rho_i - c) - \rho_i \int_0^{y^A^*(\rho_i)} F^A(\rho_i, x)dx.$$
Chapter 3. An Empirical Study

Then we calculate the expected profit for each $\rho_i$ and find the optimal price $\bar{\rho}$ by enumerating over all $\rho_i$. Then, for the additive model, the optimal price and the optimal inventory level are:

$$\bar{\rho}^A = \arg\max_{\rho_i} \{\Pi^A(\rho_i)\}$$

$$\bar{y}^A = y^A*(\bar{\rho}^A).$$

We do not exceed the observed range of prices for the search because for the unobserved prices it will not be possible to simulate the profits and make a comparison among the different policies. The drawback of limiting the prices to such a compact range is that for a number of products the optimal price of two or three of the models result in the same boundary price. For those products the simulated real profits differ only because of the different inventory policies but not the pricing policy.

By simulating the policies using the observed data we calculate the real profits which is the average of the simulated profits. Let $\bar{x}^A = \{\bar{x}^A_1, \ldots, \bar{x}^A_J\}$ be the vector of $J$ demand observations where the corresponding price is equal to $\rho_A$. For the additive model $\pi^A$ refers to the real profit which is calculated by:

$$\pi^A = \frac{1}{J}(\bar{\rho}^A - c) \sum_j \min(\bar{x}^A_j, \bar{y}^A).$$

The corresponding real profits for the multiplicative, $\pi^M$, and the general model, $\pi^G$, are calculated accordingly. In sections 3.5.1 and 3.5.2 we compare the three models based on the real profits.

The traditional way of price and inventory optimization follows a sequential approach. First, the optimal price is set, probably by the marketing department, and then the inventory decision is made given the preset price. In order to evaluate the effectiveness of this approach, we apply a sequential optimization procedure where the optimal price is calculated based on the mean demand and the inventory level is found given the optimal price, and compare this procedure with the joint optimization. Using the sequential approach the optimal price for the additive model is denoted as $\bar{\rho}^{dA}$ where $d$ stands for deterministic and the resulting real profit is $\pi^{dA}$. We compare the joint and the sequential procedures for the additive and the multiplicative models in section 3.5.3.
3.5 Simulation of profits

3.5.1 Comparison of the additive and the multiplicative models based on simulated profits

In this section we compare the additive and the multiplicative models based on the real profits. Figure 3.7 shows the histogram of profit improvement by using the multiplicative model instead of the additive where

\[
\pi^{A-M} = \frac{\pi^M - \pi^A}{\pi^A}.
\]

Note that, in the figure, we cut off the ratio at 2 and put a mass at 2 to count for the larger values. The multiplicative model seems to perform a little bit better than the additive one since \(\pi^{A-M}\) has a tendency to the positive values. When we consider all the products which satisfy the face validity conditions of both models, average \(\pi^{A-M}\) is 9.5%, and for nac. products it is 7.5%. Thus the multiplicative model results in higher profits, but only slightly and there is a considerable part (45.8%) of all products where the additive model gives higher profits.

Table 3.6 shows for each test the percentage of products in each type where the additive model results in a larger profit than the multiplicative one. Comparing the results with the PE-test for model comparison we see a slightly better performance of the linear (log-linear) model for those products,
Table 3.6: Percentage of products where the linear model performs better than the log-linear

<table>
<thead>
<tr>
<th>Type</th>
<th>Conclusion</th>
<th>$P_E$-test</th>
<th>B-P test all products</th>
<th>White test all products</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Linear</td>
<td>54.1</td>
<td>51.7</td>
<td>50.8</td>
</tr>
<tr>
<td>2</td>
<td>Log-linear</td>
<td>41.5</td>
<td>43.7</td>
<td>44.1</td>
</tr>
<tr>
<td>3</td>
<td>None</td>
<td>43.4</td>
<td>44.0</td>
<td>44.9</td>
</tr>
<tr>
<td>4</td>
<td>Both</td>
<td>46.1</td>
<td>47.1</td>
<td>46.3</td>
</tr>
</tbody>
</table>

There is also the $P_E$-test concludes that the linear (log-linear) model is the better one. This can be observed also by comparing the simulated profits with the conclusion drawn from the two tests for homoskedasticity. While there is some improvement in real profits when we use one of the methods for model selection, the value of them is marginal. The power of using any of the methods in order to select the model which results in higher profit is not satisfactory. The probability of choosing the better model is close to 50%.

The differences between the policy parameters are as expected: 60% of the time multiplicative model results in higher profits than the additive one. 30% of the time the optimal prices are the same, which might be a result of discretizing and limiting the admissible prices. Among those cases where the optimal price is the same 83% of the time optimal quantity from the additive model is larger.

3.5.2 Comparison with the general model based on simulated profits

When we compare the general model with the two classical models, we see a considerable increase in real profits. Figure 3.8 shows for all products, the profit improvement by using a general model instead of the additive and the multiplicative, $\pi^{A-G}$ and $\pi^{M-G}$, respectively. For the non-autocorrelated
### 3.5 Simulation of profits

![Graphs showing profit improvement](image)

**Figure 3.8:** Profit improvement by using the general model instead of the additive model (left) and instead of the multiplicative model (right). Last category includes ratios ≥ 2.

Products, the picture is almost the same. 82% of $\pi^{A-G}$ and 84% of $\pi^{M-G}$ are positive which means that the general model increases the profits.

The estimated mean values of $\pi^{A-G}$ and $\pi^{M-G}$ can be found in Table 3.7 with the corresponding 95% confidence interval. There is a considerable increase in profit when the general model is used. However, it should be noted that the general model that we describe and the comparison method can have some bias. Since we do not specify a model for the mean we can exactly capture the averages in the sample, but it would not be possible to estimate demand for an unobserved price. For such a purpose we need to specify a mean function which would decrease the performance of the general model. Hence not doing comparisons based on out-of sample prices creates an advantage for the general model. However, such an out-of sample comparison would be impossible because of data requirements. We have a limited number of price observations and keeping some of them out of the estimation process would not be practical. In reality even if the numbers do not turn out to be that large as reported in Table 3.7, the results show that there is potential for improvement compared to both classical models.

An interesting issue is the differences in the optimal prices and inventory levels suggested by different models. 48% of the time optimal price under the additive model is smaller than the optimal price under the general model i.e. $\hat{p}^A < \hat{p}^G$. On the other hand, $\hat{p}^M > \hat{p}^G$ 56% of the time. These numbers...
Table 3.7: Average profit improvement by using the general model (in %)

<table>
<thead>
<tr>
<th></th>
<th>All products</th>
<th></th>
<th>Nac. products</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^{A-G}$</td>
<td>47.7</td>
<td>(45.2,50.4)</td>
<td>46.4</td>
<td>(41.2,52.0)</td>
</tr>
<tr>
<td>$\pi^{M-G}$</td>
<td>47.8</td>
<td>(45.2,50.3)</td>
<td>47.3</td>
<td>(42.1,52.6)</td>
</tr>
</tbody>
</table>

are in line with the theory which says that the multiplicative model results in higher prices. When we look at the cases where $\bar{p}^A = \bar{p}^G$, we see that 94% of the time $\bar{y}^A < \bar{y}^G$. The difference is quite considerable since on average $\bar{y}^A = 0.74\bar{y}^G$. Similarly, 92% of the cases where $\bar{p}^M = \bar{p}^G$ result in $\bar{y}^M < \bar{y}^G$, and on average $\bar{y}^M = 0.67\bar{y}^G$. Therefore, both for the additive and the multiplicative models when the price is the same as the general one, the optimal quantity is larger for the general one.

3.5.3 Comparison of the joint and the sequential optimization based on simulated profits

The comparison of the sequential optimization with the joint optimization shows that there is a statistically significant improvement but the numbers are not very high. Figure 3.9 depicts the histogram of the improvement under the additive and the multiplicative demand models. For the general model it is not reasonable to calculate the optimal price in a sequential approach, since we need a function for the price dependent mean demand.

In Table 3.8 we see the confidence intervals and the average improvement. When we consider all products 4-6% profit increase can be expected by considering the stochasticity and inventory availability already for the pricing decision. However, if we combine the results from the previous section, we can say that a larger amount of improvement comes from correctly specifying

Table 3.8: Average profit improvement of using joint optimization instead of sequential optimization, under the additive and the multiplicative models (in %)

<table>
<thead>
<tr>
<th></th>
<th>All products</th>
<th></th>
<th>Nac. products</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^{dA-A}$</td>
<td>6.2</td>
<td>(4.2,8.2)</td>
<td>6.5</td>
<td>(2.9,10.4)</td>
</tr>
<tr>
<td>$\pi^{dM-M}$</td>
<td>4.1</td>
<td>(2.7,5.5)</td>
<td>2.8</td>
<td>(0.3,5.6)</td>
</tr>
</tbody>
</table>
3.5 Simulation of profits

![Graphs showing profit improvement](image)

**Figure 3.9:** Profit improvement from joint optimization under the additive model (left) and the multiplicative model (right). The last category includes ratios $\geq 2$.

the stochastic demand process. For the additive model moving from the sequential to the joint optimization brings in 6% increase, and moving from the classical to a general demand model brings in 48% increase (see table 3.7). Overall, this would mean an increase of 59% from the sequential to the joint approach with a correctly defined demand process.

The difference in the structure of the optimal policy is exactly in line with the theory: for the multiplicative model, for all cases $\bar{p}^M \leq \bar{p}^d M$ and for the additive one just 0.6% of the time $\bar{p}^A > \bar{p}^d A$ which is probably a discretization issue.

### 3.5.4 Optimal policy with a limited inventory level

In some cases there might be a fixed amount of inventory or capacity for which the price should be optimized. The revenue management literature deals with this topic considering several aspects, and there is a considerable lot of research from the classical airline to retail revenue management. Being aware of the more complex nature of this problem and more sophisticated policies, in this section we model the problem within the newsvendor framework as a single period problem. Our aim is to compare the deterministic optimal prices with the stochastic ones and gain insights about factors that effect the optimal price. In this section we do not consider the multiplicative model.
Since the inventory level is fixed, purchasing cost is not relevant and the profit is considered to be the same as revenue. When the inventory level is \( y \) we calculate the deterministic optimal price as \( \rho^d = \max(\rho_{unlim}, \rho_{lim}) \) where \( \rho_{unlim} = \arg\max_{\rho_i} (\rho_i(\hat{a} + \hat{b}\rho_i)) \) is the price which maximizes expected revenues under unlimited inventory based on mean demand, and \( \rho_{lim} = \min\{\rho_i | \hat{a} + \hat{b}\rho_i \leq y\} \) is the price which makes the expected demand equal to \( y \). For each product we consider seven inventory levels such that \( y = \text{factor} \times \rho_{unlim} \) where \( \text{factor} = \{0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4\} \).

Our main interest in this section is to gain insights into how the optimal prices are determined. Intuitively the optimal price should be where there is no gain from increasing or decreasing price for one more unit. Let us remember how the revenues are affected by price. When the price is increased from \( p_1 \) to a larger price \( p_2 \), per unit revenue would increase, and if we can still sell the same amount as we sold under \( p_2 \) revenues would increase. However, the condition of selling the same amount is not very realistic. Obviously increasing price will decrease expected demand and consequently expected sales. Hence, the increasing per unit revenue and decreasing sales should satisfy some balance at the optimal price. In order to investigate this effect further, we suggest measuring the change in expected sales by increasing price for each inventory level as

\[
\varepsilon(\rho_i, y) = -\frac{S(\rho_i, y) - S(\rho_i+1, y)}{S(\rho_i, y)} \frac{\rho_i}{\rho_i - \rho_i+1}
\]

where \( S(\rho_i, y) \) is the expected sales given \( \rho_i \) and \( y \). \( S(\rho_i, y) \) is calculated using the normal distribution resulting from the additive model. \( \varepsilon(\rho_i, y) \) measures the percentage change in expected sales by the percentage change in price, so it is the price elasticity of expected sales. When we start with a price \( \rho_i \) and increase it to \( \rho_i+1 \) if the percentage decrease in expected sales is larger than the percentage increase in price, the loss is larger than the gain and \( \rho_i+1 \) cannot be the optimal price. On the other hand, if we decrease the price to \( \rho_i-1 \) and the percentage increase in sales is smaller than the decrease in price \( \rho_i-1 \) cannot be optimal. Therefore \( \rho_i \) should give the local maximum and the change in expected sales and the change in price should be equal at \( \rho_i \), i.e. \( \varepsilon(\rho_i, y) = 1 \) at the optimal price. We know that for the additive model the optimal price is unique, so \( \varepsilon(\rho_i, y) = 1 \) should be satisfied at only one price. That would mean \( \varepsilon(\rho_i, y) \) is probably monotone increasing or decreasing in price. Let us take this discussion as our hypothesis, and try to investigate it through the observations.
### 3.5 Simulation of profits

#### Table 3.9: Price difference between the additive stochastic and deterministic models with limited inventory

<table>
<thead>
<tr>
<th>Factor</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1</th>
<th>1.2</th>
<th>1.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho^d &gt; \rho^A$</td>
<td>37.0</td>
<td>47.9</td>
<td>41.8</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\rho^d &lt; \rho^A$</td>
<td>0</td>
<td>0</td>
<td>0.2</td>
<td>29.3</td>
<td>41.5</td>
<td>33.0</td>
<td>21.1</td>
</tr>
</tbody>
</table>

We want to check if $\varepsilon(\rho^A, y) = 1$ for all quantities and products. However, since we use a discrete number of prices it is not realistic to expect that $\varepsilon(\rho^A, y) = 1$ exactly. Hence, we compare it with the elasticity of the neighbor prices in the grid. Let $\rho^A = \rho_i$, then we have two important observations:

$$\varepsilon(\rho_{i-1}, y) \leq \varepsilon(\rho_i, y) \leq \varepsilon(\rho_{i+1}, y), \quad (3.8)$$

$$\varepsilon(\rho_{i-1}, y) < 1 < \varepsilon(\rho_{i+1}, y) \quad (3.9)$$

for all the products and all the inventory levels. (3.8) means that the elasticity of sales is increasing in price at least around the optimal price, and when we look at the whole price range, we see that this continues to holds for each price, so for the additive model $\varepsilon(\rho, y)$ is monotone increasing in price for all inventory levels $y$ that we considered. Moreover, from (3.9) we conclude that $\varepsilon(\rho, y)$ is close to 1 at the optimal price.

When we consider the general model, the observations are not that exact as for the additive one. The first inequality in (3.8) is satisfied in all cases, but the second one is satisfied in only 70% of the time. Similarly, (3.9) is also not satisfied for all the cases. Hence, we can conclude that the sales elasticity is not necessarily increasing in price for the general model. However, since we do build the general model with quite loose assumptions, e.g. we do not have any assumptions on the mean. The structure of the results under this model might not be very smooth. Therefore we try to investigate the monotonicity property by fitting a linear function on the elasticities with respect to price and evaluate the slope of this function statistically. We see that 92-98% of the cases where price is significant end up with a positive price coefficient, i.e. $\varepsilon(p, y)$ increases in price.

Now we can compare the optimal prices in the light of the above discussion. Table 3.9 shows the percentage of products where the deterministic price is larger than the stochastic one and vice versa among 3223 products. For a large number of products the two prices turn out to be the same, for which we should consider the effect of limited number of feasible prices. The sales
elasticity evaluated at the deterministic price explains the differences in prices. Whenever $\rho^d > \rho^A$, $\epsilon(\rho^d, y) > 1$ so at the deterministic price $\epsilon$ is greater than 1 and in order to decrease it to 1 price should be decreased since $\epsilon$ is monotone in price. The similar argument on the other direction holds for the cases $\rho^d < \rho^A$.

When the price and inventory are jointly optimized it is well known that the stochastic optimal price is smaller than the deterministic one (see Section 2.2.3). However, we see that when the inventory level is set in advance, the relation depends on the inventory level. We can observe a structure on the number of products in each category with respect to $y$. For small inventory deterministic price is larger than the stochastic one, and for large inventory levels it is the other way around. For most of the products the prices look like as depicted in Figure 3.10.

Such a structure is not observed when we look at the difference in optimal price between the additive and the general model in Table 3.10. As inventory increases the percentage of both relations increases, which means the number of cases where the two prices are equal is decreasing in $y$. However, there is no structure with respect to the direction of the change.

Lastly, Table 3.11 shows the profit improvements by using the additive stochastic model instead of the deterministic one $\pi^{d-A}$ and using the general model instead of the additive $\pi^{A-G}$. The values are not very different than the ones in Tables 3.7 and 3.8 without the inventory limitation. While the

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig310}
\caption{Optimal price for the stochastic and deterministic models with limited inventory}
\end{figure}
3.5 Simulation of profits

Table 3.10: Price difference between the additive and the general models with limited inventory

<table>
<thead>
<tr>
<th>Factor</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1</th>
<th>1.2</th>
<th>1.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho^A &gt; \rho^G )</td>
<td>9.5</td>
<td>12.9</td>
<td>15.0</td>
<td>16.7</td>
<td>18.4</td>
<td>19.5</td>
<td>20.4</td>
</tr>
<tr>
<td>( \rho^G &lt; \rho^A )</td>
<td>29.5</td>
<td>35.8</td>
<td>41.3</td>
<td>43.7</td>
<td>46.9</td>
<td>49.1</td>
<td>50.3</td>
</tr>
</tbody>
</table>

Table 3.11: Average profit improvement of using joint optimization instead of sequential optimization under the additive model with limited inventory (in %)

<table>
<thead>
<tr>
<th>All products</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factor</td>
</tr>
<tr>
<td>( \pi^{d-A} )</td>
</tr>
<tr>
<td>( \pi^{A-G} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Nac products</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factor</td>
</tr>
<tr>
<td>( \pi^{d-A} )</td>
</tr>
<tr>
<td>( \pi^{A-G} )</td>
</tr>
</tbody>
</table>

Improvement from the deterministic to the stochastic policy is not very high, moving from the additive to the general model has a big impact. As inventory level increases, i.e. factor increases, the improvement seems to increase too. Increasing inventory might mean less restriction and more opportunity for profit improvement by changing prices.

We can summarize the findings in this chapter as follows:

1. Selection between the additive and the multiplicative models is not trivial and the selection based on some statistical criteria does not guarantee higher profits. (Table 3.6)

2. Using a more flexible model removes the selection problem and increases profit. (Table 3.7)

3. Optimal prices can be found using sales elasticity, and the sales elasticity is increasing in price for most of the products. (Section 3.5.4)
The first two observations motivate the analytical study of the price-setting newsvendor problem with a general demand definition, and the last one supports the main assumption made for the analysis of the model in the following Chapter 4.