Chapter 1

Introduction and Foundations

Inventory management and pricing decisions based on quantitative models both in industrial practice and academic works often rely on minimizing expected cost or maximizing expected revenues or profits, which refers to the concept of risk-neutrality of the decision maker. Although many useful insights in operational problems can be obtained by such an approach, it is well understood that incorporating attitudes toward risk is an important lever for building new theories in other fields, such as economics and finance. To give an example, modern portfolio theory in finance relies heavily on consideration of risk attitudes. The level of dispersion associated with an investment might be as important as the expected gain from the investment. Hence, it is necessary to find appropriate measures of risk and the appropriate objectives related to or including these risk measures.

In operations management, inventory and pricing problems especially share commonalities with the fields mentioned above. In particular, decisions have to be taken in a stochastic environment and the policy affects the risk associated with the resulting outcome. Inventory problems of their nature can be considered similar to investment problems in finance. Hence, it is important to include risk preferences in such decision problems. Moreover, this importance is supported by recent empirical findings.

In an experimental study, Schweitzer and Cachon (2000) show that for high-profit products the ordering decisions reflect risk aversion. Similarly, through an experimental newsvendor setting, Brown and Tang (2006) show that the subjects tend to order less than the expected profit-maximizing quantity because they are concerned about potential profit loss or probability of making an acceptable profit.

Besides the risk aversion of the decision makers, using expected profit as the objective implies an analytical assumption. “In many cases, the use of the expected value as an objective can be justified by the law of large numbers: if the process is repeated, the arithmetic average of the observed profits will approach the expectation.” (Collins, 2004). Specific to inventory problems, when the ordering decision is repeated many times under the very same conditions, using expected profit as the objective function can be justified. However, “if the decision is not frequently repeated or if the
outcome is large relative to wealth optimizing the expectation would not be
the appropriate objective for a risk-averse individual.” (Collins, 2004).

Following these arguments, research on risk-averse inventory models, in
particular the well-known newsvendor model with different objective func-
tions to reflect risk preferences, has become an important stream. For
example, Eeckhoudt et al. (1995) uses the concept of the expected utility
framework by modifying profit realizations with a concave increasing utility
function and Lau (1980) maximizes the probability of achieving a profit
target.

After the axiomatic foundation of coherent risk measures by Artzner et al.
(1999) the application of risk measures to inventory models became popular.
For example, in an early draft, Chen et al. (2004) uses the conditional
Value-at-Risk as objective and Jammernegg and Kischka (2007) proposes
a convex combination of low and high profits, which can be interpreted as
a mean-deviation rule. In these works results about optimal policies and
structural properties are described.

However, the different risk measures are special cases of the general class
of spectral risk measures introduced by Acerbi (2002). In our work we apply
the spectral risk measures to the inventory control and the inventory control
& pricing problem and derive optimal policies as well as structural properties.
By doing so we are able to unify the results obtained so far in the literature
under the common concept of spectral risk measures.

In the following section we introduce the newsvendor model and present
the main properties and results of the risk-neutral problem for both the
inventory and inventory & pricing problems.

1.1 The Newsvendor Model

The newsvendor model is a famous problem and building block of quantitative
inventory management. It is applicable for products with short life cycles
which become obsolete at the end of the period and cannot be stocked in order
to satisfy any demand during the next periods. Fashion apparel retailers who
must submit orders in advance of a selling season with no further opportunity
for replenishment, manufacturers who have to choose the capacity before
launch of a new product which will quickly become obsolete, or managers who
have to decide on a special one-time promotion typically face the newsvendor
problem. It also has wide applicability in service industries such as airlines
and hotels where the key decision is capacity which cannot be stored and
the product is generally perishable. The tendency towards short product
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life cycles and the growing share of service industries implies/supports the continuing interest in the newsvendor problem.

1.1.1 The inventory problem

The classical single-period, single-item, linear cost inventory control problem – the well-known newsvendor problem – is to decide on the ordering quantity before market demand is known, so that at the time of ordering demand is uncertain. The purchase cost per unit is $c$, and the product is sold to customers at a unit price $p$, which is set exogenously in the classical price-taking problem. Unsold copies can be returned to the supplier at a price $v$. To avoid trivial problem instances, it is generally assumed that $0 < c \leq p$ and $v < c$ holds.

If demand $D$, i.e. the quantity that the newsboy would be able to sell on a certain day, turns out to be equal to or greater than the ordered quantity $y$, then he makes a profit $\Pi(y, D)$ of $(p - c)y$. In the case that $D < y$ the newsvendor makes a profit of $pD + v(y - D) - cy$.

So, for a given order quantity $y$, the newsvendor’s profit $\Pi(y)$ can be written as

$$\Pi(y) = p \min(D, y) - cy + v(y - D)^+ = (p - c)y - (p - v)(y - D)^+.$$  \hspace{1cm} (1.1)

The objective in classical inventory models, i.e. models assuming a risk-neutral decision maker, is to maximize expected profit $\mathbb{E}\Pi(y)$, where we define $\mathbb{E}$ as the expectation operator. If demand $D$ were known at the time of ordering, it is easy to see that the optimal decision for the newsvendor would be to order $y^* = D$, since the function $\Pi(y)$ is a continuous piecewise linear function increasing up to $y = D$ and decreasing afterwards. However, since demand is not known at the time of ordering, the problem becomes more difficult.

The demand $D$ has to be understood as a random variable with a known demand distribution. In fact, since for real problems the exact demand distribution cannot be known either, it has to be well estimated based on collected random observations from the past. Demand can then be described by its corresponding cumulative distribution function (cdf) $F(x) := \mathbb{P}(D \leq x)$ and probability density function (pdf) $f(x)$. Since demand cannot be negative, clearly $F(x) = 0$ for any $x < 0$. 

\footnote{In the following we will omit the second argument of profit and write $\Pi(y)$ keeping in mind the dependency of profit on demand.}
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Since the average profit tends to the expected profit if the newsvendor continues his business for a long period of time, from a statistical point of view it makes sense to optimize the expected value $\mathbb{E} \Pi(y)$. Note that for simplicity $y$ is considered as a continuous rather than integer variable, which can be justified if the order quantity is reasonably large. Hence, the optimization problem can be formulated as

$$\max_y \mathbb{E} \Pi(y), \quad (1.2)$$

where

$$\mathbb{E} \Pi(y) = \int_0^y (px + v(y - x) - cy) dF(x) + \int_y^\infty (p - c)y dF(x).$$

Using integration by parts it is possible to reformulate this as

$$\mathbb{E} \Pi(y) = (p - c)y - (p - v) \int_0^y F(x) dx. \quad (1.3)$$

The function $\mathbb{E} \Pi(y)$ is concave in $y$ with a first derivative

$$\frac{d}{dy} \mathbb{E} \Pi(y) = p - c - (p - v)F(y).$$

Now let $F^{-1}(\omega)$ be the inverse function of cdf $F$, which is defined for $\omega \in [0, 1)$. Because $v < c < p$ it follows that $0 < (p - c)/(p - v) < 1$ and the optimal solution to (1.2) is

$$y^*(p) = \arg \max_{y \in \mathbb{R}^+} \mathbb{E} \Pi(y) = F^{-1}\left(\frac{p - c}{p - v}\right). \quad (1.4)$$

A more general, alternative problem formulation to (1.3), which can be commonly found in the literature, is defining marginal overage and underage cost of the order quantity. Overage cost $c_o$ is the realized cost of ordering one unit too much when demand was lower than the order quantity, while underage cost $c_u$ reflects the realized cost of ordering one unit too few for the case demand was higher than the order quantity (see e.g. Cachon and Terwiesch, 2006, for several examples).

In this work, however, we write the problem in terms of price $p$, cost $c$, salvage value $v$, and a non-negative shortage penalty cost $s$, explicitly, and do not use the model formulation based on underage and overage cost,
mainly because of notational simplicity. While the case where overage cost occur is fully equivalent with our formulation (let \( v = c - c_0 \)), the situation with shortages needs some additional consideration.

The formulation based on overage and underage cost is more general than our model assumptions as it allows to consider lost-sales as well as backordering business environments. Operating in a lost-sales business means that in a stockout situation at least the full profit margin of the product is lost. As an example, we can think of a customer entering a retailer where a certain product is out of stock. The customer does not postpone his purchase until the product is replenished but buys the product from a competitor or refrains from buying the product at all. The underage cost refers to profit margin, possibly plus some additional shortage penalties, so \( c_u \geq p - c \). This case is fully considered by our model by letting \( s = c_u - (p - c) \geq 0 \).

The backordering case, however, refers to a business where the profit margin is not (completely) lost in a stockout situation; the customer still buys the product. However, the customer might ask for some discounts to accept late delivery, or the retailer might face higher cost due to express deliveries, etc., so underage cost might not be zero. In the backordering case the relation \( 0 \leq c_u < p - c \) holds, which implies \( s = c_u - (p - c) < 0 \). In the following analysis we are not considering this case as we assume \( s \geq 0 \).

Now we are ready to extend (1.1) by shortage penalty cost and define our objective function for the risk-neutral decision maker. In the following, we also derive the optimal order quantity.

**Definition 1** (Profit function of a risk-neutral decision maker). Let \( p, c, v \) and \( s \) be the retail price (marginal revenue), product cost, salvage value and shortage penalty cost, where \( p > c > v \) and \( v, s \geq 0 \). Random demand \( D \) has a known distribution with cdf \( F \) and density \( f \). The resulting profit \( \Pi(y) \) is

\[
\Pi(y) = (p - c)D - (c - v)(y - D)^+ - (p - c + s)(D - y)^+. \tag{1.5}
\]

A risk-neutral decision maker will maximize expected profit \( \mathbb{E}\Pi(y) \) by optimizing the order quantity \( y \). This leads us to the following

**Proposition 1** (Optimal order quantity for a risk-neutral decision maker). The optimal order quantity \( y^* \) for a risk-neutral decision maker is

\[
y^* = \arg \max_{y \in \mathbb{R}^+} \mathbb{E}\Pi(y) = F^{-1} \left( \frac{p - c + s}{p - v + s} \right). \tag{1.6}
\]
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Proof. Using (1.5), the expected profit is
\[ \mathbb{E} \Pi(y) = (p-c) \mathbb{E} D - (c-v) \int_{0}^{y} (y-x) f(x) \, dx - (p-c+s) \int_{y}^{\infty} (y-x) f(x) \, dx. \]

Taking the derivative, equating to zero and solving for \( y \) leads to the well-known critical fractile solution
\[ \frac{d}{dy} \mathbb{E} \Pi(y) = -(c-v) \int_{0}^{y} f(x) \, dx + (p-c+s) \int_{y}^{\infty} f(x) \, dx \]

by Leibnitz’ rule
\[ \frac{d}{dy} \mathbb{E} \Pi(y) = -(c-v) F(y) + (p-c+s) \left( 1 - F(y) \right). \]
Hence,
\[ F(y^*) = \frac{p-c+s}{p-v+s} \]
and
\[ y^* = F^{-1} \left( \frac{p-c+s}{p-v+s} \right). \]

Important performance measures for a newsvendor from a customer’s perspective are service levels, in particular the cycle service level (CSL) and the fill rate (FR). The CSL is defined as the probability that no stockout during the selling season occurs. Sometimes called in-stock probability, it is the probability of having satisfied all demand, so the firm had stock available for each customer (Cachon and Terwiesch, 2006). This occurs if demand is not larger than the order quantity \( y \), so
\[ CSL = \mathbb{P}(D \leq y) = F(y). \tag{1.7} \]
The fill rate defined as
\[ FR = \mathbb{E} \frac{\min(D,y)}{D} \tag{1.8} \]
is the expected fraction of demand satisfied. We note here that \( FR = \frac{\mathbb{E} \min(D,y)}{\mathbb{E} D} \) can be found commonly in the literature as approximation of the fill rate (see for example in Tempelmeier, 2005).

While service levels imply customer (external) orientation as a performance measure, the probability of missing a certain profit target level \( PL_L \) is an internally oriented performance measure for the inventory problem. It can be defined as the probability that profit stays below a given level \( L \). For
example, in some managerial situations it might be important to reach a
certain target or budgeted profit level \( L \), but any overachievement does not
significantly increase utility. The performance measure \( PL_L \) now expresses
the probability that this profit target level could not be reached. We can
define the probability of missing a profit level \( L \) as

\[
PL_L := \mathbb{P}(\Pi \leq L)
\]

where in the special case \( L = 0 \) the probability of any negative profit

Further extensions to the model can be found for example in the review
of Khouja (1999), and a comprehensive presentation of the single period
problem in general can be found, for example, in Porteus (1990).

1.1.2 The inventory & pricing problem

When price is a decision variable\(^2\), single period models turn into extended
newsvendor problems. In addition to the ordering quantity, an optimal price
is set to be charged during the period. The resulting model is now more
complex because of the optimization of two variables.

The need to consider pricing and inventory problems simultaneously was
first discussed by Whitin (1955). He provides a newsvendor model with
pricing where a stochastic price-dependent demand function is assumed. He
derives an optimality condition which equates the expected loss from not
selling a marginal unit with the expected profit from selling the marginal
unit.

One of the most important issues in joint pricing and inventory models
is how to include uncertainty in the models. The common practice is to
represent the demand function as a combination of a deterministic function
and an error term. \( d(p) \) is a deterministic decreasing function of price and \( E \) is
a random variable with distribution function \( F_E(\varepsilon) \). Two typical approaches
are to combine the two terms in an additive or a multiplicative fashion.
In additive models, demand is represented as the sum of the deterministic

\(^2\)Note that we will discuss the pricing related aspects from an inventory control point
of view. This generally means that we consider simplified price-demand response
functions, while in the (empirical) marketing literature more sophisticated response
functions are used.
price dependent function and the random term, i.e. $D(p) = d(p) + E$. The corresponding expected profit is

$$
\Pi(p, y) = (p - c)y - (p - v) \int_{-\infty}^{y-d(p)} \left( y - (d(p) + \varepsilon) \right) dF_E(\varepsilon). \quad (1.10)
$$

In multiplicative models, demand is the product of the two terms, i.e. $D(p) = d(p)E$, and expected profit is

$$
\Pi(p, y) = (p - c)y - (p - v) \int_{0}^{y/d(p)} \left( y - (d(p)\varepsilon) \right) dF_E(\varepsilon). \quad (1.11)
$$

In the additive case, the mean value of the random term is generally assumed to be zero, and in the multiplicative case it is assumed to be one. Thus, for both cases, expected demand corresponds to the deterministic part, $E D(p) = d(p)$. It is common to assume $d(p) = a - bp$ with $a > 0, b > 0$ in the additive models, and $d(p) = ap^{-b}$ with $a > 0, b > 1$ in the multiplicative models (Petruzzi and Dada, 1999). It is also possible to consider any general function as long as it is decreasing in $p$.

Mills (1959) was the first to write the demand function explicitly as an additive demand model: $D(p) = d(p) + E$. The main consideration is to show the effect of uncertainty on the optimal price. For the constant marginal cost case optimal price under uncertainty is always smaller than that under certainty, but optimal ordering quantity can move in either directions. When marginal cost is increasing or decreasing optimal price may change in both directions depending on the shape of the demand curve.

Karlin and Carr (1962) also study a single period model similar to Mills (1959). However, they introduce the uncertainty in a multiplicative manner i.e. $D(p) = d(p)E$. Under this condition, the result is opposite of that under additive uncertainty. Under multiplicative uncertainty the optimal price is higher than the price under the assumption of deterministic demand. The main difference between additive and multiplicative models is the relation of price to the variance and coefficient of variation of demand. Under the additive model, while the coefficient of variation increases in price, the demand variance is constant. Under the multiplicative demand model, the coefficient of variation of demand equals that of the random term, which is independent of price, but the variance of demand is decreasing in price.

Young (1978) defined the demand function in a manner that combines both additive and multiplicative models, i.e. $D(p) = d_1(p)E + d_2(p)$. When $d_1(p) = 1$, the formulation corresponds to the additive case, and when
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d_2(p) = 0, it corresponds to the multiplicative case. Young (1978) verifies both results of Mills (1959) and Karlin and Carr (1962), and generalizes their results by describing the optimality conditions in terms of variance and coefficient of variation. However, she does not provide an explanation about the contradicting results of additive and multiplicative cases.

Petruzzi and Dada (1999) try a more integrated framework in order to provide a possible explanation of this conflict. The idea is that price is a measure to decrease the variance and coefficient of variation of demand, but it works differently for additive and multiplicative models. In the former case, “it is possible to decrease the demand coefficient of variation without adversely affecting the demand variance by choosing a lower price”; for the latter case, on the other hand, “it is possible to decrease demand variance without adversely affecting the demand coefficient of variation by choosing a higher price”. As a result, it is intuitive that the optimal price should be lower than the deterministic price in the additive model and higher in the multiplicative model.

Throughout the analysis, they use a transformation of the profit function by defining a safety factor \( z \), and describe the optimal price as a function of \( z \). For the additive case \( z = y - d(p) \), and for the multiplicative case \( z = y/d(p) \). If the realization of the random term, \( \varepsilon \), turns out to be greater than \( u \) then shortages occur, where \( s \) is the shortage cost per unit of unsatisfied demand. If \( \varepsilon \) is less than \( u \), leftovers occur.

Using a sequential approach, they first write the optimum price \( p^* \) as a function of \( z \), and solve the objective function for the optimal stocking factor \( z^* \). They find the corresponding optimal price \( p^* \) and optimal ordering quantity as \( y^* = d(p^*) + z^* \) for the additive case and \( y^* = d(p^*)z^* \) for the multiplicative case.

Yao et al. (2006) present the most general assumptions for the multiplicative and the additive models. They employ two important concepts: the price elasticity of demand and the generalized failure rate. They assume that the deterministic demand function has increasing price elasticity and that the error term has strictly increasing failure rate. Under these conditions they show that for both the additive and the multiplicative models the optimal policy is unique.

While the literature is dominated by the additive and the multiplicative uncertainty models, there are a small number of papers in which different demand models are analyzed under different approaches. Polatoglu (1991) studies a model without any assumptions on the structure of the demand-price relation and the inclusion of uncertainty. The distribution function of random demand \( D(p) \) is defined as a general price dependent function
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F(p, x). Existence and uniqueness of the optimal policy constitute the focus of the study. Kocabıyıkolu and Popescu (2009) study a general demand model without any assumption of additive or multiplicative structure, but they still use the classical definition, i.e. \( D(p) \) represents the demand as a combination of a deterministic and a random part. However, in addition to the additive and multiplicative forms, their model is also applicable for more general structures such as \( D(p) = \log(E - bp) \) or \( D(p) = \exp(E - bp) \). Their main assumption is the strict concavity of the revenue function in price for any risk realization. This assumption allows them to easily show the concavity of the profit function with respect to price and ordering quantity.

Yano and Gilbert (2004), Chan et al. (2004) and Elmaghraby and Keskinocak (2003) provide comprehensive reviews on combined pricing and inventory models both in the single-period and the multi-period settings.

1.2 Terminology, definitions used and conventions

We feel it is important to discuss and clarify the definitions of some technical terms used in this work. In particular, the term “risk” needs some further discussion since no unique definition exists in the literature. In finance literature risk generally refers to a potential loss, while classical economic theory generally deals with gains, so that risk describes a situation where gains are random variables associated with a known distribution function (cf. Müller and Stoyan, 2002, p. 265). Hanisch (2006) defines as risk of a decision alternative the possibility, that an undesired realization might occur, for example a (negative) deviation of some expected outcome\(^3\).

Hence, as the term “risk” is concerned with undesired deviations from expectation, it has to be distinguished from “dispersion”, which includes deviations in any direction. Clearly, a problem without stochasticity, i.e. without any dispersion, carries no risk; however a reduction of risk does not necessarily imply a reduction of dispersion.

The risk preference, i.e. risk-averse, risk-neutral or risk-seeking behaviour, refers to the attitude of the decision maker towards randomness. In Chapter 2 we will discuss this in detail and provide a definition of these terms.

Note that the term “risk” is sometimes used in the context of decision making to differentiate between a stochastic decision problem with full

\(^3\)In German: „Unter dem Risiko einer Handlungsalternative wird demgegenüber die Möglichkeit einer 'schlechten' Realisierung, sei es eine (negative) Abweichung von der erwarteten Entwicklung, sei es ein mit einer Alternative verbundener Verlust, verstanden.“
knowledge of the underlying distribution functions, \textit{decision making under risk}, in contrast to \textit{decision making under uncertainty} or \textit{robust decision making}, where it is not assumed that the full distribution function is known to the decision maker (cf. Schneeweß, 1967, or Laux, 2005). With respect to this classification our work falls into “decision making under risk” as we assume the distribution function of random demand to be known to the decision maker.

1.3 Structure of the work

In Chapter 2 we discuss the foundations of decision making under risk considering risk preferences of the decision maker for the case of general profit distributions. Since the main contributions to the field of risk management have been done in the field of finance and economics, most of the relevant and reviewed literature will come from that side. However, we will keep the later application to inventory control models in mind and discuss the literature in this light. As an example, while the finance literature deals mainly with loss distributions, we discuss the content based on profit distributions. A large part of this chapter will be dedicated to the conditional Value-at-Risk and its optimization, as well as the generalization of this measure to spectral risk measures.

The subsequent chapter discusses the single-period inventory control problem under consideration of risk preferences. We generalize results described in the literature so far by using the concept of spectral measures of risk for a newsvendor without incurring shortage penalty cost, as well as for the case of positive shortage penalty cost. A brief discussion on the application of risk-averse newsvendor models in the supply chain context finishes this chapter.

Chapter 4 analyzes the combined inventory & pricing problem with risk preferences. We derive optimality conditions and structural properties for the problem with zero shortage penalty cost and conduct a numerical study for the inventory & pricing problem with shortage penalty costs. The finally, Chapter 5 concludes the work.