Chapter 3

Inventory Problem with Risk Measures

While a lot of results have been obtained so far for the classical risk-neutral newsvendor problem, fewer works have considered a risk-averse or risk-seeking decision maker for the inventory problem. Early works generally cover risk preferences by applying the expected utility theory framework. Profit or losses from operations are added to the newsvendor’s final wealth. A transformation of final wealth by the utility function allows a comparison of different order quantities in a relative sense, so that an optimal order quantity can be found. Structural results of these works are not always comparable because different models are considered.

A close approach to ours of considering risk-aversion is Choi and Ruszczycyński (2008), where law invariant measures of risk\(^1\) are used. The authors find structural properties of the optimal order quantity for an inventory problem without shortage penalty cost. Their work is extended in Choi et al. (2009) to the multi-product case.

The analysis of the inventory problem depends mainly on whether or not shortage penalty cost have to be considered. It can be seen that the presence of such shortage penalty cost might result in a major change in the optimal policy. As discussed earlier, shortage penalty cost is the per-unit cost of having too few units available, which exceeds the mere lost revenues, so \(s = c_u - r > 0\). To understand the impact of penalty cost, we need to look at the relation between random demand and random profit, specifically at the correspondence between the ordered random demand and ordered profit in the sense that the \(n\)-lowest demand realization results in the \(n\)-lowest profit realization for each \(n\).

**Lemma 1.** Let \(D\) and \(\Pi\) be the random demand and profit, respectively, with realizations \(x\) and \(\pi\). Without shortage penalty cost, so \(s = 0\), there exists a one-to-one correspondence of the order of demand realizations \(x\) and the order of resulting profit realizations \(\pi\).

\(^1\)Note that the concept of law invariant measures of risk is identical with spectral risk measures.
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Proof. To show that without shortage penalty cost random profit has the same ordering as random demand, it is sufficient to see that in this case $\pi$ is (weakly) monotone increasing in $x$. Using (1.5) we can write

$$\Pi(y, x) = \begin{cases} (p - c)x - (c - v)(y - x) & x \leq y \\ (p - c)y & x > y. \end{cases}$$

(3.1)

Hence, $\Pi(y, x)$ is increasing in $x$ up to $x = y$ and then constant. □

A main consequence of Lemma 1 is that in the case $s = 0$, the operation becomes riskier the higher the order quantity is, and undesirable deviations from a certain profit level can always be reduced by ordering less initially (and hence accepting a lower expected profit). In the extreme case of not ordering at all, i.e. $y = 0$, both expected value and profit variability reduce to zero. However, once the decision maker considers shortage penalty cost, clearly this relation no longer holds. Minimizing risk now becomes a trade-off between costs resulting from overstocking and, unlike the previous case, costs from ordering too few due to shortage cost.

Hence, we will discuss the inventory problem with and without shortage penalty cost separately. In this chapter we will discuss the inventory problem with explicit consideration of risk preferences of the decision maker. While the focus is on the analysis under risk measures, in Section 3.1 we will start with a review of the existing literature on the problem, including approaches other than using risk measures, e.g. using the expected utility framework. Sections 3.2 and 3.3 are dedicated to the detailed analysis of the model with risk measures, without and with shortage penalty cost, respectively. Applications in the field of supply chain coordination and contracting under consideration of risk preferences will complete the chapter.

3.1 A review of inventory control with risk preferences

One of the early papers including risk aversion in the newsvendor context is Lau (1980), who considers three different objectives for a risk-averse newsvendor: a mean-deviation tradeoff, a utility function and the probability of achieving a minimum profit. Lau (1980) offers a formulation of the expected utility from profit with a utility function approximated by a polynomial. He gives an implicit solution for the optimal order quantity, but
it can only be solved numerically and he cannot present properties of the optimal solution.

Eeckhoudt et al. (1995) examine the newsvendor problem with more general utility functions, but their setting is not the standard newsvendor setting: after demand realization, emergency ordering is possible at a cost of \( c \leq e \leq p \). They show that risk aversion leads to lower order quantities.

Eeckhoudt et al. (1995) show:

1. If the newsvendor has decreasing absolute risk aversion, he orders more when he has larger initial wealth.

2. \( y^* \) increases in \( v \) and \( e \) similar to the risk-neutral case.

3. Changes in the cost and selling price can affect the order quantity in both directions. \( y^* \) might decrease as \( p \) increases and \( c \) decreases, which never happens in the risk-neutral case. The complication arises because of the two different effects of these parameters: the effect on the marginal benefit of \( y \) and the effect on the total wealth.

Wang et al. (2008) observed the last point shown by Eeckhoudt et al. (1995) in a numerical study and this observation caused a criticism of the expected utility theory for risk-averse newsvendor model. They observe that the newsvendor decreases his order quantity as the selling price increases, and in some cases it can be decreased almost to zero if the selling price is too high. A small degree of risk aversion for low to intermediate levels of return implies an irrationally high degree of risk aversion at the higher levels of return. While this problem was identified and treated in some fields of economics, “there is a lack of critical evaluation of expected utility theory for more complex settings such as the newsvendor problem” (Wang et al., 2008).

Keren and Pliskin (2006) solve the risk-averse newsvendor problem for a uniform demand distribution and provide a simple closed-form solution. They show that even with shortage penalty cost, a risk-averse newsvendor orders less than the risk-neutral. However, it is questionable how valuable the insights derived from using uniform distribution is since it might give counterintuitive results in case of risk aversion (see Collins, 2004).

Wang and Webster (2009) use a special form of utility functions: a piecewise linear loss aversion utility. The newsvendor considers his initial wealth as a reference level such that a final wealth below this level is considered as a loss and above it as a gain. He is more sensitive to losses
than gains but within each region the utility is linear in wealth. They show that the risk aversion may lead to higher order quantities if the shortage cost is high and the relative uncertainty of demand is low.

Lau (1980) considers a mean-deviation objective with the standard deviation of profit as the deviation measure. When there is no shortage penalty, i.e. \( s = 0 \), he shows that the risk-averse newsvendor orders less than the risk-neutral one, and he claims that the same should hold when \( s > 0 \) without giving a proof because of the complexity of the problem. However, Wu et al. (2009) state that they disprove the claim of Lau (1980) for power distributed demand. They consider the case with positive shortage cost and they use the variance of profit as the deviation measure. They show that depending on the distribution parameter, the risk-averse newsvendor might order more than the risk-neutral. Chen and Federgruen (2000) come to the same result when they formulate the objective on cost parameters.

Chen and Federgruen (2000) model the risk-averse newsvendor problem in three different ways under the mean-variance criterion. One main result of the work is: for a risk-averse newsvendor, the two objectives, profit maximization and cost minimization, might result in different decisions. They explain this difference by the dependence of revenue and cost, which yields to: \( \text{Var} (\text{Profit}) \neq \text{Var} (\text{Revenue}) + \text{Var} (\text{Cost}) \). Moreover, the decisions are different when the cost is formulated differently. They assume zero shortage cost and write the profit function for demand, \( D \), as: \( \Pi (y) = (p-c)y - (p-v)(y-D)^+ \), and additionally they write two different cost functions which are equivalent when the newsvendor is risk-neutral: \( C_1 (y) = (c-v)(y-D)^+ + (p-c)(y-D)^- \) and \( C_2 (y) = -v(y-D)^+ + p(y-D)^- + cy \). When the mean-variance rule is applied for the profit, they show that the risk aversion leads to lower order quantities if \( s = 0 \). When the objective is defined as the minimization of expected cost and the variance of cost the result is more interesting. Assuming a power demand distribution as in Wu et al. (2009), they come up with the following result: depending on the distribution parameter, risk aversion may lead to higher order quantities even without shortage cost. Moreover, the size of overage and underage costs do not necessarily have an effect on this result. Hence, the difference between risk-averse and risk-neutral decision \( y^* \) is significantly affected by the specific demand distribution.

Collins (2004) shows similar results with numerical examples using the definition of overage cost \( c_o \) and underage cost \( c_u \) (recall the discussion in Section 1.1.1). He uses the mean-variance rule for the cost but he does not specify the cost parameters in detail. He uses \( c_o \) and \( c_u \) without decomposing them into shortage cost, salvage value, etc. For gamma, negative binomial,
and normal demand distributions he shows that if the cost of underage is larger than the cost of overage, \( c_u \geq c_o \), the risk-averse newsvendor orders more than the risk-neutral one. What defines the direction of the difference between the risk-averse and risk-neutral order quantity is the relative size of \( c_o \) and \( c_u \), but not if they include shortage cost or not. Even if there are no shortage costs and no salvage value, if \( c_u = p - c \geq c_o = c \), so the profit margin is more than 50%, the risk-averse newsvendor orders more than the risk-neutral one. However, such a result is not possible when the objective is written on profit.

When the mean-variance rule is applied on cost or profit, different results come up because of the variance factor. The means are optimized at the same level, which is the solution to the risk-neutral newsvendor. However, the variance of profit and the variance of cost shows different properties. When there is no shortage cost, the distribution of profit has a bound where demand equals order quantity. For all \( D \geq y \) profit is the same, so the variance comes from the lower tail. In order to decrease this variance, the tail should be decreased which means decreasing order quantity. At the very extreme, when \( y = 0 \) there is no variance on profit. However, the distribution of cost has both tails even if \( y = 0 \). When the problem is formulated on cost it is written as \( c_o = c - v \) and \( c_u = p - c \), and each unit of demand above \( y \) costs \( c_u \), which, in fact, is an opportunity cost. This term causes the distribution of cost to become unbounded. For each \( y \) the variance of cost is proportional to \( c_0 \), which comes from the lower tail of demand, and to \( c_u \) from the upper tail, and the quantity that gives the lowest variance depends on the relation of the two parameters, independent of their decomposition.

Collins (2004) points out an important issue: how critical it is to include risk aversion in the models. He mentions that in some cases the risk-averse and the risk-neutral solutions are so close that it might not be worth it to include risk aversion in the analysis and so to deal with complicated models. Specifically, for symmetrical demand distributions, if \( c_o = c_u \) then the risk-averse and risk-neutral decisions are the same and the closer the two cost parameters are, the closer the solutions. Moreover, for the uniform distribution the two solutions are always equal independent of the difference between \( c_o \) and \( c_u \). Since a uniform distribution is commonly used for numerical examples, this result is specifically important and one should be careful about deriving general insights from these examples.

As a third approach, Lau (1980) studied the problem of maximizing the probability of achieving a profit level \( L \) with and without shortage cost. When there is no shortage cost the result is quite simple: the optimal order quantity is \( y^* = \frac{L}{p - c} \). Hence, the solution does not depend on the demand
distribution at all, since the profit has a one-to-one correspondence with demand for all \( D < y^* \), and for all \( D \geq y^* \) it stays in the targeted level \( L \). For any decrease in \( y^* \) it becomes impossible to reach \( L \), and for any increase more demand is required to cover the cost \( c \) which means a decrease in the probability.

When the shortage cost is positive, given \( y \), two different demand values can give the same profit level, so the distribution of profit is no longer a monotone function of demand. Lau (1980) presents the general solution method and gives explicit solutions for some demand distributions, such as normal and uniform distributions. Interestingly, for the uniform distribution, the optimal order quantity does not depend on salvage value at all.

From the discussion on mean-deviation objective, we know that when the problem is formulated in terms of cost, the solution becomes more complicated. Independent of shortage cost, the monotonic behaviour in demand ceases to exist and we expect that a simple solution like the one presented by Lau (1980) cannot be valid anymore.

After the axiomatic foundation of coherent risk measures by Artzner et al. (1999), the application of risk measures in inventory modeling became popular. CVaR\( _\alpha \), specifically, has become an important measure of risk in inventory modeling. Jammernegg and Kischka (2007) study the CVaR\( _\alpha \) problem focusing on the effect of risk aversion on performance measures. They formulate the objective function as a convex combination of the expected profit and the CVaR\( _\alpha \) of profit, so that they can cover both risk-averse and risk-seeking behaviour. Ahmed et al. (2007) solve the CVaR maximization problem for the newsvendor model with shortage cost. They formulate the objective in terms of cost, and the focus is on proving the existence of an optimal solution. Inclusion of shortage cost and the different formulation does not allow a simple solution, but they show that an optimum exists. Gotoh and Takano (2007) consider both the CVaR\( _\alpha \) and mean-CVaR\( _\alpha \) models with shortage cost. They use two different objectives, one formulated on profit and one on cost. In the following sections, the CVaR\( _\alpha \) and mean-CVaR\( _\alpha \) models are analyzed in more detail.

### 3.2 Basic inventory control problem

In the following section we will analyze the newsvendor's inventory problem under spectral risk measures and derive the optimality conditions. We will derive some structural properties about the optimal solution, i.e. the optimal order quantity. Further, we discuss the problem with respect to different
performance measures such as the cycle service level. After having discussed the inventory problem with general risk spectra, we will look at special cases of risk spectra such as the CVaR\(_\alpha\) or mean-CVaR\(_\alpha\) formulations as well as some continuous risk spectra such as the power and exponential function, in Section 3.2.2. We will conclude this section with a numerical analysis of the inventory problem for different formulations of the demand distribution and risk spectrum.

3.2.1 Optimal policy and structural properties for the basic inventory problem

We are now ready to apply general risk spectra to the basic inventory control problem, i.e. the newsvendor problem without shortage penalty cost.

**Proposition 4** (Newsvendor with a general risk measure). *Let the objective function of a newsvendor using a spectral risk measure be*

\[
\max_{y \in \mathbb{R}^+} M(\Pi(y)),
\]

*where*

\[
M(\Pi(y)) = (p - c)y - (p - v) \int_0^y (y - x)\phi(F(x))f(x)\,dx.
\]

*The risk measure is concave in the order quantity \(y\), and the optimal order quantity \(y^*\), is*

\[
y^* = F^{-1}\left(\Phi^{-1}\left(\frac{p - c}{p - v}\right)\right),
\]

*where \(\Phi^{-1}(\omega)\) denotes the inverse risk transformation function.*

See Appendix A for a proof.

This result shows us that the optimal order quantity of a newsvendor optimizing a measure using a general risk spectrum can be expressed in a very compact way. While for a risk-neutral newsvendor the critical fractile \(\frac{p - c}{p - v}\) refers directly to the optimal cycle service level, \(\text{CSL}^*\), a risk-averse or risk-seeking newsvendor will deviate from this solution.

Based on Proposition 4 we can obtain structural properties of the optimal solution with respect to the cost parameters as with the risk-neutral problem. From the monotonicity of \(F\) and \(\Phi\) we can immediately derive the following...
Corollary 3. The optimal cycle service level $CSL^*$ and the optimal order quantity $y^*$ are increasing in the selling price $p$ and salvage value $v$ and decreasing in cost $c$.

Note that these results are in line with the results for the risk-neutral newsvendor. In the following proposition we derive structural results about the optimal cycle service level and the optimal order quantity with respect to the risk preference using (3.4).

Proposition 5. For a newsvendor without shortage penalty cost, the optimal cycle service level $CSL^*$ and the order quantity $y^*$ increase in the risk preference $\eta$.

**Proof.** From Definition 6 we know that $\Phi$ decreases in $\eta$, and its inverse $\Phi^{-1}$ increases in $\eta$. It follows immediately that $y^*$ increases in $\eta$. \(\square\)

This result generalizes results previously described in the literature. For a newsvendor applying a CVaR$_\alpha$ objective function, Chen et al. (2004) show that the optimal order quantity for a risk-averse newsvendor will not exceed the risk-neutral optimal order quantity. For different formulations of mean-deviation rules similar results were found. Jammernegg and Kischka (2007) find that the optimal order quantity is increasing in $\alpha$ (i.e. decreasing in the level of risk aversion) and decreasing in $\lambda$ (higher values of $\lambda$ puts higher weights on the $\alpha$-quantile and lower weights on the expected value and, hence, imply a higher level of risk aversion).

The intuition behind this behaviour is clear: Since the newsvendor incurs no shortage penalty cost other than lost profit $p - c$, the risk in profit comes only from unsold leftover inventory. Hence, by reducing the order quantity, the newsvendor can always reduce his risk by accepting the reduced expected profit. Later in Section 3.3 we will see that in the case of positive shortage penalty cost this is no longer true.

Continuing the discussion about non-strict ordering of $\Phi$ from Section 2.3.1, where we gave an example of two intersecting risk transformation functions in Figure 3.1, we can see now that in this case the order of the optimal cycle service level and order quantity depends on the critical fractile. For smaller ranges of the target cycle service level, decision maker $DM_1$ will order more than decision maker $DM_2$ and act less risk-averse; for larger ranges this relation will change. While for some ranges of the critical fractile, e.g. for low selling prices, decision maker $DM_1$ orders more than $DM_2$, but as the critical fractile increases this relation changes.
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Figure 3.1: Two decision makers with different risk spectra: $DM_1$ with CVaR$_\alpha$ risk spectrum, $\alpha = 0.5$ (solid line), $DM_2$ with mean-CVaR$_\alpha$ risk spectrum, $\alpha = 0.1$, $\lambda = 0.5$ (dashed line). Note that the optimal cycle service level is the transformed critical fractile. For a low price $p_1$ $DM_1$ orders more than $DM_2$, while for a high price $p_2$ this relation changes.

**Proposition 6.** The expected profit $\mathbb{E} \Pi$ of a newsvendor using a spectral measure of risk is maximized for the risk-neutral optimal order quantity. $\mathbb{E} \Pi$ decreases the more the newsvendor’s risk preference deviates from risk neutrality.

**Proof.** Recall that $\mathbb{E} \Pi$ is concave in $y$, with its maximum in the risk-neutral optimal order quantity. From Proposition 5 we know that $y^*$ increases in $\eta$. Hence, $\mathbb{E} \Pi$ is unimodal in the risk preference. □

This result was already described by Jammernegg and Kischka (2007) for a newsvendor using a mean-CVaR$_\alpha$ risk spectrum (see the discussion later in Example 6).

While the cycle service level can be seen as an external, i.e. customer oriented performance measure, the probability of missing a profit target level as defined in (1.9) is an internally oriented performance measure.

**Lemma 2.** Let $L$ be a given profit target level. The probability of missing the profit level $L$ is the probability that profit is below the target. For a newsvendor without shortage penalty cost

$$PL_L := \mathbb{P}(\Pi \leq L) = \begin{cases} F\left(\frac{(c-v)y+L}{p-v}\right) & \text{for } L \leq (p-c)y \\ 1 & \text{for } L > (p-c)y. \end{cases} \quad (3.5)$$

Hence, in the case of $y \geq \frac{L}{p-c}$, $PL_L$ is monotone increasing in $y$.

See Appendix A for a proof.

Note that in the special case $L = 0$ the probability of any negative profit realization is considered (cf. Jammernegg and Kischka, 2007).
Proposition 7. Let a newsvendor with a general admissible risk spectrum order \( y^* \) such that it maximizes his risk measure \( M(y) \). \( PL_L \) increases in the risk preference for any \( L \leq (p - c)y \) and is 1 otherwise.

**Proof.** Using Proposition 5, the optimal order quantity \( y^* \) increases in \( \eta \). From Lemma 2, \( PL_L \) increases in \( y \) for \( L \leq (p-c)y \) and is 1 otherwise. Hence, \( PL_L \) increases in the risk preference for \( L \leq (p-c)y \) and is 1 otherwise. □

This result was already found by Jammernegg and Kischka (2007) for a mean-CVaR\(_\alpha\) decision maker for the case \( L = 0 \). In this case the optimal probability of missing the profit target is increasing in \( \alpha \) and decreasing in \( \lambda \). Very related to Proposition 7 is Lau (1980), who describes a situation where a budgeted profit might be established such that a manager may be interested primarily in maximizing the probability of reaching this budget. In this case it might be less important if the limit is strongly exceeded or just reached. Lau (1980) derives the optimal order quantity which we can formulate in the following:

**Corollary 4 (Maximizing the probability of reaching a profit target).** The optimal order quantity \( y^* \), which maximizes the probability of reaching a certain profit target level \( L \), is

\[
y^* = \frac{L}{p-c}
\]

On this result Lau (1980) comments “This result is somewhat strange: to maximize the probability of attaining \( L \), one sets the decision variable such that \( L \) is also the largest possible profit attainable.” However, we think this result is quite intuitive: since the profit distributions are ordered with respect to \( y \) up to \( (p-c)y \), once the newsvendor already ordered \( y^* = \frac{L}{p-c} \) there is no benefit from ordering a single unit more, since the maximum potential profit increase is not considered, while the worst profit outcome, \(-cy\), is decreasing. If the newsvendor orders one unit less than (3.6), he cannot reach the profit target level at all.

Let us state here that the primary use in this work of the probability of missing a profit target level is as performance measure, not as objective function. Hence, the result of Corollary 4 is mainly presented for the sake of completeness. Among the recent works, Shi and Chen (2007) consider maximizing the probability of reaching a profit target level in a supply chain context, and in Shi et al. (2010) a combined inventory and pricing approach is taken.
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3.2.2 Specific examples of risk spectra in the basic inventory problem

Using some examples, we will now show how the general spectral measure can be used to solve the inventory problem for already-known formulations, such as the expected value solution, the CVaR$_{\alpha}$ objective function or a mean-deviation objective where deviation is expressed by CVaR$_{\alpha}$. As all of these formulations are special cases of spectral measures, the examples illustrate the flexibility of using general risk spectra in the problem analysis.

**Example 4** (Expected value). A first example is the expected value formulation derived in (1.4). Using the expected value implies neutrality about the variability of the outcome, hence the spectral function does not assess higher weights to lower outcomes. The risk spectrum is

$$\phi(\omega) = 1. \quad (3.7)$$

It can be easily seen that applying (3.7) to (3.4) results in the optimal order quantity for a risk-neutral newsvendor

$$y^* = F^{-1}\left(\frac{p-c}{p-v}\right)$$

as previously stated in (1.4).

**Example 5** (CVaR$_{\alpha}$). As we discussed earlier in Section 2.3 when describing the general spectral measure, CVaR$_{\alpha}$ is another special case where the risk spectrum is a constant function $\frac{1}{\alpha}$ in the range $0 \ldots \alpha$ as in (2.10), so that the risk transformation function is

$$\Phi(\omega) = \begin{cases} \omega \frac{1}{\alpha} & \omega \leq \alpha \\ 1 & \text{otherwise}. \end{cases} \quad (3.8)$$

Its inverse is then

$$\Phi^{-1}(t) = \alpha t \quad \text{for } t \in [0, 1] \quad (3.9)$$

so that the optimal order quantity is

$$y^* = F^{-1}\left(\alpha \cdot \frac{p-c}{p-v}\right). \quad (3.10)$$
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This result can be found for example in an early draft of Chen et al. (2004), who, in their work, specifically used the \( \text{CVaR}_\alpha \) objective function to model risk-averse decision making behaviour.

An observation we can make for a \( \text{CVaR}_\alpha \) spectrum is that the optimal cycle service level is limited by \( \alpha \). If we consider \( DM_1 \) in Figure 3.1 again (the solid line), then it is easy to see that the maximum cycle service level \( CSL_{\text{max}} = \alpha \) is reached for \( \Phi = 1 \), so for \( \lim_{p \to \infty} \frac{p-c}{p-v} \). This implies that independent of the profitability of the product, the optimal order quantity will not exceed a certain amount, \( F^{-1}(\alpha) \).

This implies, for the distribution of profit, that the optimal order quantity will be such that \( \text{VaR}_\alpha(\Pi(y^*)) = (p-c)y^* \) (see Gotoh and Takano, 2007, or Chen et al., 2008b). The intuition behind this is that a \( \text{CVaR}_\alpha \) decision maker only considers profit outcomes below \( \text{VaR}_\alpha \); any realization above \( \text{VaR}_\alpha \) is not considered. Hence, using the fact that the maximum possible profit realization under the “best” demand state is \((p-c)y\), there is no benefit from ordering any quantity where the resulting \( \text{VaR}_\alpha(\Pi(y)) \) is smaller than \((p-c)y\).

Example 6 (Mean-CVaR\( \alpha \) objective). The mean-CVaR\( \alpha \) objective function is a special case where the risk spectrum is a piecewise constant function with a single step in \( \alpha \) as in (2.11). Recall that this risk spectrum has two parameters. It is composed out of a \( \text{CVaR}_\alpha \) part with parameter \( \alpha \) and the expected value in a convex combination, where the weighting factor is \( \lambda \). Its risk transformation function is

\[
\Phi(\omega) = \begin{cases} \frac{\omega}{\alpha} & \omega \leq \alpha \\ \lambda + \frac{1-\lambda}{1-\alpha}(\omega - \alpha) & \text{otherwise.} \end{cases} \tag{3.11}
\]

The inverse risk transformation function in this case is

\[
\Phi^{-1}(t) = \begin{cases} \frac{\alpha}{\lambda} t & t \leq \lambda \\ \alpha + \frac{1-\alpha}{1-\lambda}(t - \lambda) & \text{otherwise.} \end{cases} \tag{3.12}
\]

Plugging (3.12) in (3.4) leads to the optimal order quantity

\[
y^* = \begin{cases} F^{-1} \left( \frac{\alpha}{\lambda} \frac{p-c}{p-v} \right) & \frac{p-c}{p-v} \leq \lambda \\ F^{-1} \left( \frac{p-c}{p-v} + \frac{1-\lambda}{1-\lambda} \frac{c-v}{p-v} \right) & \text{otherwise.} \end{cases} \tag{3.13}
\]

A mean-CVaR\( \alpha \) objective of this type was already proposed and the optimal policy found in Jammernegg and Kischka (2007) and Gotoh and Takano.
(2007). While the latter are mainly interested in finding a linear programming formulation for solving the capacity constraint multi-product case, Jammernegg and Kischka (2007) derive structural properties for the risk-averse \((\alpha < \lambda)\) as well as for the risk-seeking \((\alpha > \lambda)\) behaviour. They show that the optimal order quantity \(y^*\) and the optimal cycle service level \(CSL\) are increasing in \(\alpha\) and decreasing in \(\lambda\), hence decreasing in the level of risk aversion.

Note that as long as the critical fractile \(\frac{p-c}{p-v}\) is smaller than \(\lambda\), the solution of the mean-CVaR\(_\alpha\) optimizer is not different from the pure CVaR\(_\alpha\) optimizer.

Further, Jammernegg and Kischka (2007) point out that expected profit decreases the more risk-averse or risk-seeking the decision maker becomes. This result can be obtained considering the fact that the optimal order quantity of a risk-neutral decision maker results in the maximum expected profit, and by the monotonicity of the order quantity in the risk preference.

3.2.3 Numerical study of the basic inventory control problem

In this section we will present a numerical study in order to illustrate the findings introduced in the previous section and will discuss some of the structural properties of the problem in more detail.

For numerical analysis of the basic inventory control problem we use the following parameters: selling price \(p = 10\), production cost \(c = 6\), salvage value \(v = 3\), no shortage penalty costs are considered (see Section 3.3.3 for numerics of the inventory problem with positive shortage penalty cost). Additionally, for the mean-CVaR\(_\alpha\) risk spectrum, whenever it is not stated otherwise, we set \(\lambda = 0.5\).

To model demand uncertainty, in the following we assume two parametric distributions of demand.

1. A Weibull distribution \(D \sim \text{Weib}(2, 100)\) is used as a general rule, since it is shown to be a “Newsvendor distribution” by Braden and Freimer (1991). The expected demand, \(\mathbb{E}D = 88.62\) units, the optimal risk-neutral cycle service level is \(CSL^* = 0.5714\) and the corresponding optimal order quantity is \(y^* = F^{-1}(0.5714) = 92.05\) units.

2. A Gamma distribution \(D \sim \text{Gamma}(\mu, \sigma^2)\) will be used for all cases where the effect of demand variance is of interest, since the Gamma distribution allows for changing variance while keeping the mean constant, which is not possible for the Weibull distribution. Expected
demand is again $\mathbb{E} D = \mu = 88.62$; $CSL^*$ and $y^*$ depend on the actual variance.

Note that both distributions have a positive support which fits with the assumption of a non-negative demand $D$, unlike the commonly used normal distribution.

We use the following definitions of both distribution functions. We define the Weibull distribution with two parameters, the shape $\gamma$ and scale $\delta$ parameter, as

$$F(x) = 1 - e^{-\left(\frac{x}{\delta}\right)^\gamma},$$

with corresponding density,

$$f(x) = \frac{\gamma}{\delta} \left(\frac{x}{\delta}\right)^{\gamma-1} e^{-\left(\frac{x}{\delta}\right)^\gamma}.$$  \hspace{1cm} (3.14)

For the Gamma distribution we use as parameters mean $\mu$ and variance $\sigma^2$ directly, so that we can modify them independently. Hence, we define the cdf as

$$F(x) = \frac{\gamma \left(\frac{\mu^2}{\sigma^2}, \frac{x\mu}{\sigma}\right)}{\Gamma\left(\frac{\mu^2}{\sigma^2}\right)},$$

where here $\Gamma$ is defined as the complete Gamma function and $\gamma$ is the lower incomplete Gamma function\(^2\). All numerics are calculated using the $R$ language and environment for statistical computing (R Development Core Team, 2010).

**Results from the numerical study**

As stated in Proposition 5, $y^*$ is monotone in the level of risk aversion for the inventory problem without shortage penalty cost. Figure 3.2 shows the monotonicity of $y^*$ in the level of risk aversion for a mean-CVaR$_\alpha$ and a power risk spectrum. Note that in the mean-CVaR$_\alpha$ model, for a fixed $\alpha$, $y^*$ is decreasing in $\lambda$, which can be seen from the perfect ordering of the $y^*$ lines in the first plot of Figure 3.2.

\(^2\)The Gamma function is defined as $\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} \, dt$. The lower incomplete gamma function is defined on the same integrand, $\gamma(z, r) := \int_r^\infty t^{z-1} e^{-t} \, dt$. Note that there exist efficient numerical approximations for the (incomplete) Gamma function as well as for the probability and density of the Gamma distribution in the $R$ environment for statistical computing.
### 3.2 Basic inventory control problem

In Figure 3.2 we can see that the risk preference can have a big impact on $y^*$. For example, with the power risk spectrum, $y^*$ changes between 20 and 120. However, the size of the difference depends on the parameters of the demand distribution. Figure 3.3 shows the effect of demand variance on $y^*$. The lines are ordered with respect to $\eta$ and they get more and more distant as variance increases. Clearly, risk preference is an issue of uncertainty or randomness, and if we see variance as a measure of uncertainty, it is obvious that large variance causes a more significant impact of the risk preference on the decision.

In Figure 3.4 we illustrate the expected profit evaluated at $y^*$ with respect to the risk preference obtained by maximizing the risk-measure for a mean-CVaR$_\alpha$ and a power risk spectrum. Clearly, the expected profit is maximized in the risk-neutral case and decreases when the decision maker deviates from risk neutrality, becoming more risk-averse or more risk-seeking. Hence, there are some profit levels smaller than the risk-neutral expected profit which can be reached by an order quantity which is optimal for a specific risk-averse, as well as for a risk-seeking, decision maker.

Figure 3.5 considers a 90% confidence interval of profit, $\text{CI} \Pi$, and the maximum and minimum possible profits, $\pi_{\text{max}}$ and $\pi_{\text{min}}$, respectively. For most of the $\alpha$-levels and all $k$ the upper border of the confidence interval

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**Figure 3.2:** Optimal order quantity $y^*$ for a mean-CVaR$_\alpha$ (left plot) and a power (right plot) risk spectrum objective function. Note that the circles on the lines correspond to the risk-neutral $y^*$ and for each line separate between risk-averse (left) and risk-seeking (right) behaviour.
Figure 3.3: Optimal order quantity $y^*$ for a mean-CVaR$_\alpha$ (left) and power (right) risk spectrum as a function of demand variance $\sigma^2$, where $D \sim \text{Gamma}(\mu, \sigma^2)$.

Figure 3.4: Expected profit with corresponding $y^*$ for a mean-CVaR$_\alpha$ (left plot) and a power (right plot) risk spectrum with demand $D \sim \text{Weib}(2,100)$. Note that the vertical line separates risk-averse ($\alpha < \lambda$ and $k < 1$) from risk-seeking ($\alpha > \lambda$ and $k > 1$) behaviour.
3.3 Inventory control with shortage penalty cost

We will now discuss the case where the newsvendor considers shortage penalty cost in addition to lost revenues in case of a stockout situation for each unit short. Similar to the basic inventory problem without shortage cost we will formulate the problem for a general risk spectrum and illustrate properties for specific examples afterwards.

Figure 3.5: Expected profit with corresponding $y^*$ for a mean-CVaR$_{\alpha}$ (left plot) and a power (right plot) risk spectrum with a 90% confidence interval CI $\Pi$ and maximum/minimum profit $\pi_{\text{max}}$ and $\pi_{\text{min}}$, respectively.

corresponds to $\pi_{\text{max}}$. Note that the size of the confidence interval, as well as the range $\pi_{\text{max}} - \pi_{\text{min}}$, is increasing in $\eta$.

In Figure 3.6 the probability of missing different profit targets is shown. As the left plot shows, $PL_L$ is 1 for order quantities below $(p - c)y$ and increasing in $y$ for any quantity larger than this threshold. It can be easily seen that $PL_L$ is ordered with respect to the target level $L$ in the sense that the higher the profit target level, the larger $PL_L$ is. The right plot shows $PL_L$ at $y^*$ with respect to the level of risk aversion, specifically with respect to $k$ for a power risk spectrum. The structure shown in the left plot is almost directly carried on to the right one because of the monotonicity of $y^*$ in $k$. 

3.3 Inventory control with shortage penalty cost

We will now discuss the case where the newsvendor considers shortage penalty cost in addition to lost revenues in case of a stockout situation for each unit short. Similar to the basic inventory problem without shortage cost we will formulate the problem for a general risk spectrum and illustrate properties for specific examples afterwards.
Chapter 3. Inventory Problem with Risk Measures

3.3.1 Optimal policy and structural properties for the inventory problem with shortage penalty costs

Once the newsvendor considers shortage penalty cost for unsatisfied demand, the ordering of profit is not the same as of demand anymore, as discussed in Lemma 1. If we consider for now the maximum possible profit realization for a given quantity $y$ as the profit target, so $\Pi_L = (p-c)y$, then a deviation of this profit target can happen now for a demand realization larger than $y$ (i.e. profit loss due to understocking) in addition to the realizations smaller than $y$ (i.e. profit loss due to overstocking).

Recall that the newsvendor’s profit function with shortage penalty cost can be written as

$$\Pi(y) = \begin{cases} (p-c)D - (c-v)(y-D) & D \leq y \\ (p-c)y - s(D-y) & D > y, \end{cases}$$

(3.17)

with cdf $F_{\Pi}(\pi) = \mathbb{P}(\Pi(y) \leq \pi)$.

In Figure 3.7 the profit distribution for a newsvendor without and with shortage penalty cost are shown. The cdf of profit without shortage penalty cost has a jump in $(p-c)y$ to 1, since there is a probability mass on this profit realization. For the case with shortage penalty cost such a point does not exist, and therefore the profit cdf is continuous. Further, without shortage penalty cost, the minimum possible profit is $-(c-v)y$ and the profit
3.3 Inventory control with shortage penalty cost

Figure 3.7: Cumulative distribution function of profit without considering (left), and while considering (right) shortage penalty cost with parameters $p = 10$, $c = 6$, $v = 3$; $s = 0$ (left) and $s = 5$ (right).

distribution has a limited support to the left. With shortage penalty cost the minimum possible profit is $-\infty$ with demand $\infty$, so the profit distribution has unlimited support on the left tail.

Figure 3.8 depicts profit with respect to demand for a given quantity, $y$. Note that the maximum possible profit is $\pi_{\text{max}} = (p - c)y$. Let us for now consider any profit $\pi_1 \in [-(c - v)y, (p - c)y)$. Each profit in this range will happen with exactly two different demand realizations; with demand $x_1 < y$ and with a corresponding demand $\bar{x}_1 > y$. Equating both cases of (3.17) for $\pi(x,y)$, we can express $\bar{x}$ in terms of $x$, so

$$\bar{x} = y + (y - x)\frac{p - v}{s}. \quad (3.18)$$

Note that the corresponding upper demand level $\bar{x}$ for a given $x$ depends on $y$. This becomes immediately clear from Figure 3.8 if we keep $x$ constant and increase $y$. In this case $\pi_{\text{max}}$ increases. Since the slopes of both parts of this piecewise linear function stay the same ($p - v$ for $x \leq y$ and $-s$ for $x > y$), $\bar{x}$ is necessarily increasing.

From Figure 3.8 we can see that the probability that profit is below $\pi_1$ is composed of two parts: (a) the probability that demand is below $x$, and (b) the probability that demand is larger than $\bar{x}$. From (3.18) we can express $\bar{x}$ as a function of $x$, so we are able to express the sum of the
Chapter 3. Inventory Problem with Risk Measures

Figure 3.8: Profit $\Pi$ as a function of random demand $D$. Maximum profit $\pi_{\text{max}}$ is reached for $D = y$. If no shortage penalty cost are considered ($s = 0$), $\Pi$ remains constant for any $D > y$, while for the case with shortage penalty cost ($s > 0$), $\Pi$ decreases in $D$. Note that in the latter case, each profit realization above $-(c - v)y$ can happen with exactly two different demand levels.

two probabilities in terms of $x < y$ as long as we also consider negative realizations of demand. Profit realizations below $-(c - v)y$ can only occur in the case where demand $D > y \left(1 + \frac{p - v}{s}\right)$, which would correspond to $x < 0$. Hence, if we only consider the demands in the range $(-\infty, y]$ or alternatively, $[y, \infty)$, we are able to cover all possible profit realizations. In the following analysis we will use the range $(-\infty, y]$, which implies considering negative demand realizations $x < 0$, although they happen with probability 0.

Lemma 3 (Profit distribution). Let $\Pi$ be random profit with distribution $F_\Pi(\pi)$ and $D$ random demand with distribution $F(x)$. There exist two mutually exclusive demand levels, $x \in (-\infty, y]$ and $x \in (y, \infty)$ where profit equals $\pi$. Hence,

$$F_\Pi(\pi) = \begin{cases} F\left(\frac{\pi + (c - v)y}{p - v}\right) & \text{for } \pi \leq (p - c)y \\ 1 - F\left(\frac{(p - c + s)y - \pi}{s}\right) & \text{for } \pi > (p - c)y, \end{cases}$$

(3.19)

with density

$$f_\Pi(\pi) = \begin{cases} \frac{1}{p - v} f\left(\frac{\pi + (c - v)y}{p - v}\right) + \frac{1}{s} f\left(\frac{(p - c + s)y - \pi}{s}\right) & \text{for } \pi \leq (p - c)y \\ 0 & \text{for } \pi > (p - c)y, \end{cases}$$

(3.20)
See Appendix A for a proof. Since we formulated the profit distribution for $x < y$ in Lemma 3, by (3.17) $x$ is such that $\pi(x, y) = (p-c)x - (c-v)(y-x)$. Plugging in $\pi(x, y)$ in (3.19),

$$F_{\Pi}(\pi(x, y)) = F(x) + 1 - F\left(y + (y-x)\frac{p-v}{s}\right).$$

Further, since we previously defined $\bar{x} := y + (y-x)\frac{p-v}{s}$,

$$F_{\Pi}(\pi(x, y)) = F(x) + 1 - F(\bar{x}). \quad (3.21)$$

Note that (3.21) forms a new distribution function for any $x \leq y$. Considering (3.18), in the following we call the new distribution function $G(x) := F_{\Pi}(\pi(x, y)) = F(x) + 1 - F(\bar{x})$ with density $g(x) := f(x) + f(\bar{x})\frac{p-v}{s}$. Note that even if we do not write it explicitly, the order quantity $y$ is a parameter of the distribution function $G(x)$.

In light of the above discussion we are now ready to formulate the risk measure in the following proposition.

**Proposition 8 (Newsvendor with a general risk measure).** Let the objective function of a newsvendor using a spectral risk measure be

$$\max_{y \in \mathbb{R}^+} M(\Pi(y)) = (p-c)y - (p-v) \int_{-\infty}^{y} G_{\Phi}(x) \, dx \quad (3.22)$$

where

$$M(\Pi(y)) := (p-c)y - (p-v) \int_{-\infty}^{y} G_{\Phi}(x) \, dx \quad (3.23)$$

and

$$G_{\Phi}(x) := \Phi(G(x)) = \Phi\left(F(x) + 1 - F(\bar{x})\right).$$

The risk measure is concave in the order quantity $y$. Furthermore, we can write the optimal order quantity $y^*$ as the solution to the first order condition,

$$M'(\Pi(y^*)) = \frac{dM(\Pi(y))}{dy}\bigg|_{y=y^*} = 0, \quad (3.24)$$
where the first derivative of the risk measure with respect to the order quantity \( y \) is

\[
M'(y) = \int_{-\infty}^{y} \left[ -(c - v)g(x)\phi(G(x)) \right] dx \\
+ \int_{-\infty}^{y} \pi(x, y)\frac{dg(x)}{dy} \phi(G(x)) dx \\
+ \int_{-\infty}^{y} \pi(x, y)g(x)^2 \phi'(G(x)) dx \\
+ (p - c)yf(y)\phi(1)\left(1 + \frac{p - v}{s}\right) .
\] (3.25)

See Appendix A for a proof.

The concavity result follows from the general results obtained by Acerbi (2002) stated in Proposition 3(a). As mentioned earlier, \( G(x) \) forms a new distribution function on demand for a specific order quantity, so that for each \( y \) a different \( G \) exists. For a given \( y \), one can think of \( G \) as cdf of the sum of two exclusive, conditional random demands. Hence, when shortage penalty costs are considered, we can transform the demand distribution \( F \) into a distribution \( G \) such that the same ordering of profit and demand exists. Important, however, is that now we have to consider negative demand realizations. Although having a probability of zero, each of them has a corresponding positive demand realization \( x > y \), where the newsvendor incurs shortage penalty costs.

Instead of explicitly formulating \( M(\Pi) \) for the case of a risk-averse decision maker, one could also use the maximization formula by Acerbi (2002) directly. However, to be able to solve (2.15) numerically, the risk spectrum has to be discretized into a piecewise linear function with a resolution of \( J \) steps. This discretized \( \phi \) can be used for an optimization using (2.18). However, the complexity of the optimization problem grows in the number of steps since the optimization has to be carried out over \( (y, \psi^J) \), or \( J + 1 \) variables. Additionally, for the case of a risk-seeking decision maker a joint optimization on \( (y, \psi) \) does not seem possible anyway.

Using (3.23) together with (3.25) allows for using highly efficient single dimensional numerical optimization algorithms. The reason why we are able to reduce the \((J + 1)\)-dimensional optimization problem to a single dimensional one is that we are taking advantage of the specific ordering of profit realizations, while Acerbi’s method does not assume any knowledge
on the ordering of profit. A more detailed discussion about the derivation can be found in the proof of Proposition 8 in Appendix A.

3.3.2 Specific examples of risk spectra in the inventory problem with shortage penalty cost

As we did in the previous section for the basic inventory control problem, we can now look at special cases of risk spectra already used in the literature and describe the optimal policies and structural properties found so far. Note again, that for the risk-neutral case, i.e. for the expected value optimization, \( \phi(\omega) = 1 \), the transformations described in the previous section are not necessary. When all random realizations are given the same weight, the different ordering of the demand realizations compared to profit realizations is not relevant. It is easy to find the optimal order quantity \( y^* \) for this case as 

\[
y^* = \frac{p - v}{p - v + s} F^{-1} \left( \frac{p - c + s}{p - v + s} \right); \quad \text{see the discussion of the risk-neutral problem in the introduction in Section 1.1.1.} \]

In the following, we will give as examples the pure CVaR\( _\alpha \) optimizer and a mean-CVaR\( _\alpha \) optimizer.

**Example 7** (CVaR\( _\alpha \) decision maker with shortage penalty cost). Unlike the general formulation of the optimal order quantity in 3.25 the problem can be solved for the optimal order quantity \( y^* \) in closed form in the following

**Lemma 4** (Optimal order quantity for a CVaR\( _\alpha \) optimizer with shortage penalty cost).

\[
y^* = \frac{p - v}{p - v + s} F^{-1} \left( \alpha \frac{p - c + s}{p - v + s} \right) + \frac{s}{p - v + s} F^{-1} \left( 1 - \alpha \frac{c - v}{p - v + s} \right).
\]

See Appendix A for a proof.

While this explicit formulation specifically for the CVaR\( _\alpha \) decision maker of the optimal policy was previously derived by Gotoh and Takano (2007), we present a proof based on the optimization of (3.22). Their original proof is based on the CVaR\( _\alpha \) optimization by Rockafellar and Uryasev (2000) and solves the problem for both \( y^* \) and optimal Value-at-Risk \( \psi^* \). Note that the case without shortage penalty cost is a special case of (3.26); as for \( s = 0 \), the right term vanishes and the weighting factor for the first term becomes 1, so that the whole equation reduces to the solution of the CVaR\( _\alpha \) solution without shortage penalty cost in (3.10).

\(^{3}\)Note that Acerbi (2002) calls the problem of not knowing the ordering of profit with respect to the state variable “reshuffling” of profit.
Example 8 (Mean-CVaR$_\alpha$ decision maker with shortage penalty cost).
Extending the results of the CVaR$_\alpha$ decision maker of the previous example leads to the result of a mean-CVaR$_\alpha$ decision maker.

Lemma 5 (Optimal order quantity for a mean-CVaR$_\alpha$ optimizer with shortage penalty cost).  The optimal order quantity $y^*$ is that $y$ which solves the following system of equations

$$\frac{\lambda}{\alpha} \left( (p - c + s)(1 - F(x^o)) - (c - v)F(x^o) \right)$$

$$+ \frac{1 - \lambda}{1 - \alpha} \left( (p - c + s)F(x^o) + (c - v)F(x^o) - (p - v + s)F(y) \right) = 0 \quad (3.27)$$

$$F(x^o) + 1 - F(\bar{x}^o) = \alpha. \quad (3.28)$$

The proof is omitted here and provided with Example 9 where a general piecewise constant risk spectrum is considered.

It seems to be impossible to find an explicit formulation for $y^*$ for general demand distributions, although for specific families of distributions an explicit solution can be obtained by plugging in the distribution function. Solving both equations numerically is highly efficient compared to applying numerical optimization algorithms on $M(\Pi(y))$ as defined in (3.23) directly, since no numerical integration has to be carried out.$^4$

Example 9 (General piecewise constant risk spectrum with shortage penalty cost).  The results of the mean-CVaR$_\alpha$ example can be further generalized by considering a piecewise constant risk spectrum, e.g. to discretize a continuous risk spectrum in a piecewise constant function with $J$ jumps as shown and discussed in Section 2.3.3. The risk spectrum in $w$ was already shown in Figure 2.5, while the corresponding inventory problem is illustrated in Figure 3.9. If $J$ jumps exist, then there are $J + 1$ levels of $\phi$, so $\phi_1 \ldots \phi_{J+1}$. The demands where the jumps occur are then $x_1 \ldots x_J$. Further, we define $x_0^o := -\infty$ and $x_{J+1}^o := y$, i.e. the smallest and largest demand realization in the demand range of interest. Recall from the discussion in Section 3.3.1 that considering the demand range $(-\infty, y]$ is enough to cover all possible profit realizations.

$^4$The computation of the distribution function might require numerical integration techniques, if no closed-form expression exists. An example for such a cdf is the Gamma distribution. In those cases, generally efficient approximations implemented in numerical software packages (e.g. R) exist.
3.3 Inventory control with shortage penalty cost

$$\phi$$

Figure 3.9: A discretized risk spectrum with J jumps in the probability range, so that J + 1 levels of $\phi$ exist. The profit levels at the borders of each range $i$ are $\Pi_i = -\infty, \pi_1, \ldots, \pi_J, (p - c)y$, with corresponding demand levels below $y$ of $D = -\infty, x_1^\circ, \ldots, x_J^\circ, y$.

Lemma 6 (Optimal order quantity of a newsvendor with piecewise constant risk spectrum considering shortage penalty cost). Let $x_i^\circ$ be the demand where

$$F(x_i^\circ) + 1 - F(\bar{x}_i^\circ) = \omega_i,$$  \hspace{1cm} (3.29)

so that $x_i^\circ$ are the demands in the range $(-\infty, y]$ up to which point the profits are weighted with $\phi_i$. We can formulate $M$ using a piecewise constant $\phi$ as

$$M(\Pi(y)) = \sum_{i=1}^{J+1} \phi_i \int_{x_{i-1}}^{x_i} \left((p - c)x - (c - v)(y - x)\right)g(x)dx.$$  \hspace{1cm} (3.30)

The optimal order quantity $y^\ast$ satisfies the following system of equations (i.e. the first order condition):

$$\frac{dM(\Pi(y))}{dy} = 0 =$$

$$= \sum_{i=1}^{J+1} \phi_i \left((p - c + s)(F(\bar{x}_{i-1}^\circ) - F(\bar{x}_i^\circ)) - (c - v)(F(x_i^\circ) - F(x_{i-1}^\circ))\right),$$

$$F(x_i^\circ) + 1 - F(\bar{x}_i^\circ) = \omega_i \hspace{1cm} \text{for all } i = 1 \ldots J. \hspace{1cm} (3.31)$$

See Appendix A for a proof.

Note that for J jumps a system of $J + 1$ nonlinear equations has to be solved.
Chapter 3. Inventory Problem with Risk Measures

3.3.3 Numerical study of the inventory control problem with shortage penalty cost

For the numerical analysis of the inventory control problem with shortage penalty cost we use the same set of parameters as with the numerics for the basic inventory problem in Section 3.2.3. Additionally, we assume a shortage penalty cost $s = 5$ unless otherwise noted.

Figure 3.10 shows the optimal order quantity in the level of risk aversion. Unlike the case of zero shortage penalty cost, now $y^*$ is no longer monotone in the level of risk aversion. While the order quantity increases as the decision maker becomes risk-seeking, the order quantity is not non-increasing as the decision maker becomes more risk-averse. For some ranges of risk-aversion the quantity is reduced. However, as the decision maker becomes very risk-averse (the risk preference is extremely low), her focus turns towards reducing the impact of the very rare case where demand is extremely high and high shortage penalty cost are realized. Hence, the order quantity increases again.

Note that $y^*$ goes to infinity as $\alpha$ or $k$ gets closer to zero. In Figure 3.10 this property is not very clear, but if we increase the penalty cost to $s = 30$ this effect becomes more clear, as can be seen in Figure 3.11. Hence, we see that a risk-averse decision maker might order more than a risk-neutral one.
3.3 Inventory control with shortage penalty cost

Figure 3.11: Optimal order quantity $y^*$ for a mean-CVaR$_{\alpha}$ (left plot) and a power (right plot) risk spectrum objective function with large shortage penalty cost, $s = 30$.

Figure 3.12: Optimal order quantity $y^*$ for a mean-CVaR$_{\alpha}$ (left) and power (right) risk spectrum as a function of demand variance $\sigma^2$, where $D \sim \text{Gamma}(\mu, \sigma^2)$. 
Chapter 3. Inventory Problem with Risk Measures

The effect of demand variance on \( y^* \) is shown in Figure 3.12. Similar to the case without penalty cost (see Figure 3.3), the difference between \( y^* \) with respect to risk aversion gets larger as variance increases. Note that for a risk-averse newsvendor, i.e. \( \alpha = 0.2 \) or \( k = 0.5 \), if there is no shortage penalty cost, \( y^* \) is decreasing in variance (see Figure 3.3), while now, when \( s = 5 \), it is increasing. However, this is not a general rule but depends on the cost parameters. For example, if the shortage penalty cost is decreased to \( s = 3 \), \( y^* \) is again decreasing in variance for \( \alpha = 0.2 \) and \( k = 0.5 \) as in the case of zero penalty cost.

In Figure 3.13 we illustrate the expected profit for a mean-CVaR\( _\alpha \) and a power risk spectrum. The expected profit is maximized in the risk-neutral case and decreases once the decision maker becomes more risk-seeking. For the case of risk aversion, however, the expected profit is no longer monotone since the optimal order quantity is not monotone in the level of risk aversion as shown in Figure 3.10.

Figure 3.14 shows the 90% confidence interval and the maximum possible profit at \( y^* \) with respect to the level of risk aversion. In any case, the minimum possible profit is \(-\infty\), and this causes the confidence interval to be larger compared to the zero penalty cost case. Specifically, for small \( \alpha \) and \( k \), when \( s = 0 \) the newsvendor is able to significantly decrease the difference between \( \pi_{\text{max}} \) and \( \pi_{\text{min}} \) by ordering very little and consequently achieving quite a tight CI (see Figure 3.5). However, here when \( s > 0 \), even

![Figure 3.13: Expected profit with corresponding \( y^* \) for a mean-CVaR\( _\alpha \) (left plot) and a power (right plot) risk spectrum with shortage penalty cost.](image-url)
3.3 Inventory control with shortage penalty cost

Figure 3.14: Expected profit with corresponding $y^*$ for a mean-CVaR$_\alpha$ (left plot) and a power (right plot) risk spectrum with a 90% confidence interval CI $\Pi$ and maximum profit $\pi_{\text{max}}$. Note that minimum profit is $-\infty$.

when the newsvendor is very risk-averse, he is not able to reach such a small confidence interval.

Figure 3.15 depicts the probability of missing different profit target levels. Note that unlike the case without shortage penalty cost, $PL_L$ has no jump at $(p-c)y$ anymore but is a continuous function. When using the optimal order quantity, the shape of $PL_L$ is influenced by the shape of the optimal order quantity in the level of risk aversion as shown in Figure 3.10. The non-monotonicity of $y^*$ causes the increasing-decreasing-increasing shape of $PL_L$, e.g. for $L = 150$. 
Figure 3.15: Probability of missing a profit target $L$, i.e. $P(\Pi \leq L)$ with different order quantities (left) and with the degree of risk aversion $k$, where the corresponding optimal order quantity for a power risk spectrum is used.

3.4 Applications in supply chain management

So far in this chapter, we have discussed the inventory control problem from the view of a single decision maker, as when the newsvendor is a single entity or an agent in a supply chain. The problem can also be viewed from a supply chain perspective. When risk sensitivity is included in supply chain coordination issues, the complications are twofold: the optimal policies and coordinating contracts become more complicated, and the objective of the whole chain gets more difficult to describe. “When each of the agents maximizes his expected profit, the objective of the supply chain considered as a single entity is unambiguously to maximize its total expected profit (…). Regardless of the measure used, when one or more agents in the supply chain are risk-averse, it is no longer obvious as to what the objective function of the supply chain entity should be.” (Gan et al., 2004).

Lau and Lau (1999) and Tsay (2002) focus on return policies concerning a single manufacturer and a single retailer who are both risk-averse. Lau and Lau (1999) assume both parties have mean-variance objective functions and all of the leftover inventory on the retailer’s side can be returned to the manufacturer, so the policy parameter is just the salvage value and not the proportion of leftovers that can be returned. They obtain the optimal wholesale price and salvage value for normally distributed demand, but the optimality refers to maximizing the manufacturer’s objective function. They
show that as the manufacturer becomes more risk-averse, he sets both the salvage value and the wholesale price lower, which basically means that he tries to put more of the risk on retailers’ shoulders. Since the problem is modeled from the side of the manufacturer the supply chain performance is not taken into account.

Tsay (2002) assumes a single manufacturer, single retailer setting, both maximizing their mean-standard deviation objective. The manufacturer sets the return policy and then the retailer sets his selling price after the uncertain demand is revealed. The demand distribution is modeled as a two-point distribution, i.e. low with probability \( \rho \), high with \( 1 - \rho \). The policy is either a no-returns policy or a full-return for full-credit policy, so not a continuum as in Lau and Lau (1999). They find the equilibrium under each policy. In case there are no returns the manufacturer’s risk sensitivity has no effect since there is no uncertainty on his side. When the retailer orders too few because of his risk aversion, the manufacturer lowers the wholesale price to increase the retailer’s order. As a result, depending on the demand parameters, the retailer’s risk aversion might cause his expected profit to increase because of the decreasing wholesale price. In case of full-return for full-credit, a retailer’s risk sensitivity has no effect on the policy parameters, while depending on the degree of his risk aversion the manufacturer may lower the wholesale price when he accepts returns.

When the manufacturer is risk-neutral he increases the wholesale price if he accepts returns. However, returns make his profit more variable and one way to decrease variability is by decreasing the variability on sales. Hence, if he is risk-averse he lowers the wholesale price, which induces lower retail prices and smaller variability in profit. Hence, all else equal, the retailer should search for a risk-averse supplier.

Agrawal and Seshadri (2000) assume a risk-neutral manufacturer selling items to a number of newsvendors who differ in their risk sensitivity, which is measured by a mean-variance rule. Newsvendors operate in identical and independent markets and the selling price of the product is the same in each market. The manufacturer does not know the degree of risk aversion of each single newsvendor, but he knows the distribution of risk aversion among them. Under this setting they design a menu of contracts which should be offered by a risk-neutral intermediary who bares the risk of the newsvendors in different proportions depending on the contract that the newsvendor selects from the menu. Each contract in the menu includes a risky part which comes from the uncertain profit and a fixed side-payment from the intermediary to the newsvendor. The more risk-averse newsvendors select one of the contracts with a large side-payment. Chen and Seshadri (2006)
prove the optimality of Agrawal and Seshadri’s menu of contracts in the sense that it maximizes the intermediary’s return, which means designing a setting that increases the order quantities to the optimal level in the risk-neutral case. However, as mentioned by the authors, if pricing is also considered, the existence of an intermediary might cause higher retail prices and lower consumption.

While Lau and Lau (1999) and Tsay (2002) do not mention coordination at all, Agrawal and Seshadri (2000) consider the total order quantity in the channel but none of them considers Pareto optimality.

Gan et al. (2004) are the first to examine coordinating contracts based on Pareto optimality with risk-averse agents. Their definition of supply chain coordination assumes “no agent’s payoff can be improved without impairing someone else’s payoff and each agent receives at least his reservation payoff.” They differentiate between the channel’s external and internal problem as the order/production quantities, and the allocation of profit. When there is at least one risk-neutral agent within the supply chain, as a Pareto optimal sharing rule, he can take all the risky profit and give side-payments to the other agents when the external decision is set to maximize the chain’s expected profit. This statement is in line with the results of Agrawal and Seshadri (2000).

In order to develop coordinating contracts Gan et al. (2004) consider a supply chain with a single retailer and a single manufacturer. When both agents maximize their mean-variance tradeoffs, or an exponential utility function, the revenue-sharing contract and the buy-back contract can coordinate the chain if a side-payment to the retailer is included. If the manufacturer is risk-neutral it is Pareto optimal if he bears all the risk and just gives a fixed payment to the retailer. This result is extended by Chen et al. (2008a).

However, the results cannot be generalized to a concave utility function. They give an example where the manufacturer is risk-averse at low returns and risk-neutral at higher levels. For this specific example they show that neither the buy-back nor the revenue sharing contract can coordinate the channel since it is not possible to develop a proportional sharing rule. They mention that for general cases new contract forms should be designed.

Gan et al. (2005) study a supply chain with a risk-neutral manufacturer and a retailer who has a constraint on the probability of reaching a certain profit. The standard revenue-sharing and buy-back contracts do not coordinate the channel anymore. They construct a coordinating contract which is quite complicated compared to the contracts for expected profit maximizing agents.
3.4 Applications in supply chain management

Chen et al. (2008a) study a decentralized supply chain with multiple risk-neutral or risk-averse agents. They introduce the concept of rational contracts and analyze a supply chain with multiple risk-averse suppliers and a single risk-averse retailer. For the second case, the authors identify conditions of coordinating contracts and propose specific contracts based on the level of risk aversion among the suppliers and the retailer. Chen et al.'s contract includes fixed side-payments as mentioned by Gan et al. (2004) and also the concept of intermediaries mentioned by Agrawal and Seshadri (2000). They show that if the level of risk aversion is the same between all the players in the supply chain, any contract that coordinates the risk-neutral case coordinates this case as well. If the retailer and the manufacturers have different levels of risk aversion the type of the coordinating contract changes depending on who the least risk-averse player is.

Wang and Webster (2007) study supply chain coordination contracts between a single risk-neutral supplier and a single risk-averse retailer using a piecewise linear utility function as already discussed in Wang and Webster (2009). Their results indicate that coordinating contracts based on the assumption of risk neutrality may result in markedly lower supply chain profit when retailers are loss-averse; hence, suppliers should consider the impact of loss aversion in contract design, in particular when dealing with small retailers for whom the assumption of risk neutrality is less likely to hold.

Lastly, two papers on newsvendor networks by Tomlin and Wang (2005) and van Mieghem (2007) include risk aversion in the network design problem. Tomlin and Wang (2005) consider unreliable resources and uncertain demand. They show that for a risk-averse decision maker, dedicated sourcing may be more preferable than the flexible one. The only uncertainty in van Mieghem (2007) is the demand uncertainty and he shows that the risk-averse newsvendor may increase network capacity more than the risk-neutral one.

The main conclusions of the papers presented in this section are: when the agents with different degrees of risk aversion require different contracts, some of the coordinating contracts assuming risk neutrality do not work under risk aversion, and one way of dealing with this problem is introducing risk-neutral intermediaries into the channel. In the end, when risk aversion is considered, the contracts and/or the design of the supply chain become more complicated.