Appendix B

Financial Constraints in Perfect Capital Markets

This section provides an example of financial constraints in perfect capital markets under Walrasian equilibrium theory to intertemporal decisions of economic agents. Despite the fact that the specific forms of financial constraints in intertemporal maximization models depend on assumptions, model structures, etc., they are all based on the concept of intertemporal solvency of agents which can be best illustrated by the following simple example\(^1\) of an intertemporal maximization problem in an open economy with a free and perfect international capital market, reading as: Find a sequence of consumption levels \(C_t, C_{t+1}, \ldots\) that maximizes utility \(U_t\) of a representative agent over an infinite planning horizon lasting from date \(t\) to \(T \to \infty\), being specified by

\[
U_t = \sum_{s=t}^{\infty} \beta^{t-s} u(C_s), \quad 0 < \beta < 1, \tag{B.1}
\]

subject to the two constraints

\[
\sum_{s=t}^{\infty} \left( \frac{1}{1 + i_r} \right)^{s-t} C_s + \lim_{T \to \infty} \left( \frac{1}{1 + i_r} \right)^T F_{t+T+1} = (1 + i_r)F_t + \sum_{s=t}^{\infty} \left( \frac{1}{1 + i_r} \right)^{s-t} Y_s, \tag{B.2}
\]

\[
\lim_{T \to \infty} \left( \frac{1}{1 + i_r} \right)^T F_{t+T+1} \geq 0, \tag{B.3}
\]

where \(\beta\) denotes the subjective discount or time-preference factor being constant over time, \(i_r\) the real interest rate for borrowing or lending in the (perfect) international capital market being also assumed to be constant over time, \(Y_s\) income at the end of period \(s\), \(C_s\) consumption at the end of period \(s\), \(F_t\) the value of the economy's initial net foreign assets at time \(t\), and \(\lim_{T \to \infty} (1 + i_r)^{-T} F_{t+T+1}\) the discounted present value of the economy's net foreign assets at the end of the (infinite) planning horizon. Note that net foreign assets in any period \(s\), \(F_s\), can also take a negative sign, i.e. \(-F_s\), denoting net foreign debt in period \(s\). The change in net foreign assets over period \(s\), i.e. \(F_{s+1} - F_s\), is equivalent to the current account over period \(s\), \(CRA_s\), being defined as

\[
CRA_s = F_{s+1} - F_s = Y_s - C_s + i_r F_s, \tag{B.4}
\]

\(^1\)This example is a modified version of Obstfeld and Rogoff (1999), pp. 715-721.
i.e. as the difference between income and consumption in period $s$, plus interest income on net foreign assets in case $F_s > 0$, or less interest payments on foreign debt in case $F_s < 0$ in period $s$; note that constraint B.2 is obtained by rearranging the current account identity B.4.

The maximization problem B.1 is subject to the two intertemporal financial constraints B.2 and B.3, where constraint B.2 is the intertemporal budget constraint, having being derived, as aforementioned, from the current account identity B.4, and B.3, the inequality constraint, the no-Ponzi-game condition. Regarding constraint B.2, the expression $\sum_{s=t}^{\infty} (1 + i_r)^{t-s} (Y_s - C_s)$ represents the present value of aggregate savings, being equivalent to the present value of the trade balance and thereby to the present value of net output the economy transfers to foreigners during the planning period from $t$ to $T \rightarrow \infty$. As a result, the present value of net wealth of the domestic economy at time $t$, $PNW_t$, is composed of the initial value of foreign assets or debt $(1 + i_r)F_t$, and the present value of aggregate savings during the planning period, i.e. formally it holds that

$$PNW_t = (1 + i_r)F_t + \sum_{s=t}^{\infty} (Y_s - C_s).$$  \hspace{1cm} (B.5)$$

Equation B.5 corresponds, according to constraint B.2, to the discounted present value of net foreign assets or debts at the end of the planning period in $T \rightarrow \infty$, i.e. it holds that

$$PNW_t = \lim_{T \rightarrow \infty} \left( \frac{1}{1 + i_r} \right)^T F_{t+T+1}$$ \hspace{1cm} (B.6)$$

where $PNW_t$ and $\lim_{T \rightarrow \infty} (1 + i_r)^{-T}F_{t+T+1}$ can be positive, zero or negative.

In case the present value of net wealth is positive or zero, i.e. in case there exist net foreign assets or no foreign assets at the end of the planning period ($F_{t+T+1} \geq 0$), being formally described by

$$PNW_t = \lim_{T \rightarrow \infty} \left( \frac{1}{1 + i_r} \right)^T F_{t+T+1} \geq 0, \text{ implying } -(1 + i_r)F_t \leq \sum_{s=t}^{\infty} (Y_s - C_s),$$ \hspace{1cm} (B.7)$$

then, the domestic economy is said to be \textit{intertemporally solvent}. By way of contrast, if the present value of net wealth is negative, i.e. if there exists net foreign debt at the end of the planning period ($F_{t+T+1} < 0$), being formally described by

$$PNW_t = \lim_{T \rightarrow \infty} \left( \frac{1}{1 + i_r} \right)^T F_{t+T+1} < 0, \text{ implying } -(1 + i_r)F_t > \sum_{s=t}^{\infty} (Y_s - C_s),$$ \hspace{1cm} (B.8)$$

then, the domestic economy is said to be \textit{intertemporally insolvent}.

The intertemporal solvency conditions B.7 and B.8, having been derived by wealth considerations above, are a combination of the two financial constraints B.2 and B.3 of the initial maximization problem. As a result, the intertemporal solvency condition B.7 can be alternatively derived by inserting the no-Ponzi-game condition B.3 into the intertemporal budget constraint B.2, which results in the \textit{intertemporal solvency} condition

$$-(1 + i_r)F_t \leq \sum_{s=t}^{\infty} \left( \frac{1}{1 + i_r} \right) (Y_s - C_s),$$  \hspace{1cm} (B.9)$$
implying \( \lim_{T \to \infty} (1 + i_r)^{-T} F_{t+T+1} \geq 0 \). Intertemporal solvency arises, according to equation B.9, either by a positive present value of the trade balance which is greater than or equal to the initial value of foreign debt (where \( F_t < 0 \), implying that \(-(1 + i_r) F_t > 0\)), or by a negative present value of the trade balance which is in absolute terms smaller than or equal to the initial value of net foreign assets (where \( F_t > 0 \), implying that \(-(1 + i_r) F_t < 0\)).\(^2\) Accordingly, intertemporal insolvency arises in case the present value of what the economy produces exceeds or is equal to the present value of its consumption, i.e. in case the present value of the trade balance is positive or zero.\(^3\)

By way of contrast, the intertemporal insolvency condition B.8 can be obtained in case the no-Ponzi-game condition B.3 is violated, i.e. in case

\[
\lim_{T \to \infty} \left( \frac{1}{1 + i_r} \right)^T F_{t+T+1} < 0. \tag{B.10}
\]

As a result, insertion of the violated no-Ponzi-game condition B.10 into the intertemporal budget constraint condition B.2 yields the intertemporal insolvency condition, being formally given by

\[
-(1 + i_r) F_t > \sum_{s=t}^{\infty} \left( \frac{1}{1 + i_r} \right) (Y_s - C_s). \tag{B.11}
\]

Accordingly, intertemporal insolvency can arise, according to inequalities B.11 and B.8, either from a positive present value of the trade balance which is smaller than the initial value of foreign debt (where \( F_t < 0 \), implying that \(-(1 + i_r) F_t > 0\)), or from a negative present value of the trade balance which is smaller than initial value of net foreign assets (where \( F_t > 0 \), implying that \(-(1 + i_r) F_t < 0\)).\(^4\) Both cases imply that the domestic economy is intertemporally insolvent since the present value of net wealth \( PNW_t \) is negative, corresponding to a violation of the no-Ponzi-game condition B.3, i.e. formally in both cases it holds that

\[
PNW_t = \lim_{T \to \infty} \left( \frac{1}{1 + i_r} \right)^T F_{t+T+1} < 0.
\]

Accordingly, intertemporal insolvency arises in case the present value of what the economy consumes exceeds the present value of its output by a positive amount which never converges to zero, i.e. in case the present value of the trade balance is negative.\(^5\) As a

\(^2\)Another, but trivial condition for intertemporal insolvency arises in case of a positive present value of the trade balance being combined with initial net foreign assets, resulting in a steady increase in net foreign assets.

\(^3\)This statement may be confusing because it has been stated that intertemporal solvency can also arise in case of a negative present value of the trade balance which is however smaller than the initial value of net foreign assets. However, the initial value of net foreign assets can be interpreted as an initial trade balance surplus which is larger than the present value of future trade balance deficits which results in an overall positive present value of the trade balance.

\(^4\)As it was the case with intertemporal solvency, another, but also trivial condition for intertemporal insolvency arises in case of a negative present value of the trade balance which is combined with initial foreign debt.

\(^5\)Note that an initial value of debt can be interpreted as an initial trade balance deficit which can be larger than the present value of future trade balances surpluses, resulting in an overall negative present value of the trade balance.
result, in case the domestic agents "play" a Ponzi-game, the domestic economy borrows continually to meet interest payments on foreign debt, which grows at least at the international interest rate $i_r$, and does not repay debt by reducing consumption below income, i.e. by trade balance surpluses. From the viewpoint of foreign creditors, the domestic economy is intertemporally "overindebted" or bankrupt due to the inability to repay debt which means a costless transfer of resources from foreigners to domestic agents.

The solution of the maximization problem B.1 requires, according to constraints B.2 and B.3, that the domestic economy has to remain intertemporally solvent either by a positive stock of net foreign assets or by zero net foreign assets at the end of the planning period. However, assuming a positive stock of net foreign assets at the end of the planning period would imply a costless transfer of resources of the domestic economy to foreigners, or, to put it differently, bankruptcy of the foreign economy. As a result, in order to rule out domestic as well as foreign bankruptcy, implying that all resources are used up at the end of the planning period, the no-Ponzi-game condition B.3 has to be modified into the transversality condition

$$\lim_{T \to \infty} (1 + i_r)^{-T} F_{t+T+1} = 0,$$

which is one in four conditions which are necessary and sufficient for optimality. The transversality condition implies that only the equality version of the intertemporal solvency condition B.9, i.e.

$$-(1 + i_r)F_t = \sum_{s=1}^{\infty} \left( \frac{1}{1 + i_r} \right) (Y_s - C_s),$$

(B.13)

can guarantee an optimum. The necessary first-order condition for maximizing $U_t$ with respect to consumption is given by the intertemporal Euler-equation

$$u'(C_s) = (1 + i_r) \beta u'(C_{s+1}),$$

(B.14)

where $u'(C_s)$ denotes marginal utility from consumption in period $s$, and $(1 + i_r)\beta u'(C_{s+1})$ marginal utility from future consumption in $s+1$ by reducing period $s$ consumption and lending it for one period which is converted into $(1 + i_r)$ units of period $s+1$ consumption raising utility in period $s$, $U_s$, by $(1 + i_r)\beta u'(C_{s+1})$. The Euler equation states that at utility maximum, both quantities have to be equal, i.e. that the consumer cannot gain from feasible shifts of consumption between periods.\(^6\) Thus, for the present maximization problem, conditions B.14, B.4 (or equivalently B.13), and B.12 are necessary and sufficient for optimality.

\(^6\)A special case arises if it holds that $\beta = (1 + i_r)^{-1}$, i.e. in case the subjective discount factor equals the market discount factor, which leads to the modified Euler condition $u'(C_s) = u'(C_{s+1})$, stating that the consumption path is flat due to the wish to smooth consumption over the entire lifetime.