3. Methods of extracting business cycle characteristics

Before one can study the business cycle, the latter has to be extracted from the underlying time series. For that purpose, at least a vague idea of the nature of the business cycle is necessary.

3.1 Defining the business cycle

From studying the literature it becomes apparent, that there is no consensus about the nature and definition of the business cycle. This lack of a precise definition results first of all from the absence of a widely accepted theory of the business cycle phenomenon. The most prominent approaches relate to demand-side imbalances caused by information asymmetries between economic agents or to supply-side shocks due to technical progress (real business cycle theories). These problems make theory a poor guide for measuring business cycles. Secondly, the empirical approach for discriminating between competing theories is problematic, too. Looking at the behaviour of economic time series, one can see smooth up and down changes (especially if one imagines some kind of simple trend) without any regularities concerning the length or amplitude of a cycle. As a consequence, there exists a plethora of statistical tools and methods for extracting this kind of movements.

3.1.1 The classical business cycle definition

Despite a long tradition in business cycle analysis, starting from the middle of the last century by Burns - Mitchell (1946), their definition still forms the basis of one strand of business cycle studies:

"Business cycles are a type of fluctuations found in the aggregate economic activity of nations that organise their work mainly in business enterprises: a cycle consists of expansions occurring at
about the same time in many economic activities, followed by similarly general recessions, contractions and revivals which merge into the expansion phase of the next cycle; this sequence of changes is recurrent but not periodic; in duration business cycles vary from more than one year to ten or twelve years; they are not divisible into shorter cycles of similar characters with amplitudes approximating their own."

As such a definition is far too imprecise for being a working base for a study; it has been the subject of several refinements and extensions. This definition, also called the classical definition of the business cycle, has been criticised not only for being imprecise concerning e.g. what constitutes the aggregate economic activity and the pattern of these movements, but also for being "measurement without theory." In order to study the classical business cycle, several economic time series deemed representing the aggregate economic activity are analysed, with trended and stationary time series (e.g. interest rates) both being observed within the same framework. It is not required to separate stationary movements from a trend, only some prior adjustment for working days and seasonal variations is recommended. But, without breaking down changes in economic activity into trend growth and cyclical movements it may be difficult to interpret the cycle, with problematic consequences for economic policy interventions.

Apart from shortcomings in the theoretical definition, this method does not distinguish between different sources of economic growth and has lost some of its popularity. If this definition of

---

23 Examples for this are Zarnowitz (1992), and Zarnowitz – Moore (1982).


25 See Koopmans (1947).

26 Whereas business cycle analysis with level data lost its scientific importance in the last decades, there are some recent studies by Harding – Pagan (2002), Hess – Iwata (1997) and Clements – Krolzig (2004) giving evidence for a kind of revival of such approaches.
business cycle is applied where the focus lies on developments in the level of time series, economies showing a strong trend growth will experience only few recessions.

3.1.2 The deviation cycle definition

As business cycles seem to be recurrent but not periodic, as Burns – Mitchell (1946) pointed out, Mintz (1969) suggested separating the trend from time series in order to study them properly. This was quite in line with Lucas' (1977) suggestion of defining the business cycle as the deviations of aggregate real output from trend, which puts the definition on a more theoretical basis. The Solow (1970) growth model, describing production as following a secular trend – mostly driven by technological progress – with demand-driven imbalances causing temporary departures (the business cycle) from it, serves here as the theoretical framework. This is probably the definition most people have in mind, when they talk about the business cycle.

If the business cycle is defined as departure from a trend it is called the "deviation cycle" or "growth cycle" approach. As the term "growth" usually refers to growth rates or first order differences in logs – which represents a special method for de-trending time series and therefore a kind of deviation cycle approach- the term "deviation cycle" will always be used henceforth.

The choice of the appropriate definition of business cycles – classical or deviation cycle – may depend on the specific research topic. If the purpose is to give guidance for business cycle intervention (as deviations from the steady state growth path imply welfare losses) then the deviation cycle is more appropriate. This is based on the fact, that the business cycle defined by the devia-

---

27 Sometimes "deviation cycles" and "growth cycles" are used as synonyms; in this case, calculating first differences for extracting the trend is labelled the "growth rate cycle" approach in order to distinguish between them.
tion concept shows a lead in time vis-à-vis the one interpreted in the classical way. Furthermore, it has to be kept in mind that applying the classical approach yields fewer and shorter contraction phases than the deviation method as documented in Harding – Pagan (2002). This may lead to a substantially different business cycle stabilisation policy. Especially the lead property makes deviation cycle approaches more appropriate for economic policy issues. For all those reasons, the following analysis concentrates exclusively on the deviation cycle approach.

3.2 Isolation of business cycle frequencies

Following Zarnowitz (1992), economic time series measuring output may be represented in the following form

\[ Y_t = X_t + C_t + S_t + e_t \]  

where \( X_t \) represents the trend component, \( C_t \) the business cycle variation and \( S_t \) other components like seasonal, working day or weather effects. \( e_t \) represents an error term capturing all residuals like special events and measurement errors. In earlier approaches, all components have been modelled as being deterministic, whereas modern ones understand all or some of them as stochastic processes.

\[ F \text{ Frequently this model is specified in logarithms where the underlying model is multiplicative } Y_t = X_t \cdot C_t \cdot S_t \cdot e_t. \]

\[ G \text{ Examples for this kind of models are structural time series models following the approach of Harvey (1989) where the trend is modelled as a stochastic process. Also the method of using just growth rates (out of first order differencing process) which implies random walk behaviour of the trend component can be regarded as a stochastic approach.} \]
Even the assumption of orthogonality of trend and cycle has been relaxed by some studies\textsuperscript{30}, which is in line with Zarnowitz' (1992) consideration that trend and cycle are possibly influencing each other.

As theory specifies neither the trend nor the business cycle precisely, there is a large dissent in business cycle research, how the business cycle can be extracted from the underlying time series. As a consequence, several methods for isolating the cyclical component have been proposed.

Based on the component structure given in Figure 1, business cycle extraction methods can be classified mainly by three none mutually exclusive characteristics

- direct or indirect
- univariate or multivariate
- model-based or heuristic (filter) techniques

Direct approaches try to extract directly only cyclical variations (according to a specified definition) out of one or more time series, without prior adjustment of other components like trend, seasonality or noise. This class comprises univariate filter techniques like band-pass filters as well as models assuming a special structure for all components\textsuperscript{31}. The latter can be constructed in a multi- or univariate manner.

Indirect approaches by contrast, first try to single out other information not belonging to the business cycle in order to obtain the business cycle as the residual. As the data used here are already cleared for seasonal and working day effects, this is mainly a task of separating a trend from the series, eventually followed by some

\textsuperscript{30} Delias (2003) estimated a business cycle model where trend and cycle interact in a non-trivial way.

\textsuperscript{31} See Harvey (1989).
smoothing procedure to get rid of high frequency noise variation. The subtraction of deterministic time trends, first order differencing, the Hodrick-Prescott filter, the Christiano-Fitzgerald filter and some ad-hoc moving averages belong to this class. In most cases, several steps have to be combined in order to single out the business cycle.

Model-based approaches estimate all or some components by assuming some specific structure for them. Further model-based approaches concentrate on time series models. As they do not assume a specific structure, they can also be regarded as filter techniques. This goes for instance for the Beveridge-Nelson decomposition where the economic time series is represented as a time series model which is factorised after identification in order to extract the business cycle.

3.2.1 Outliers

A proper identification of the business cycle requires a consistent data base, adjusted for disturbances caused by outliers. This goes for all approaches, whether they are model- or filter-based, direct or indirect methods. According to the decomposition possibilities outlined above, these outliers are contained in most cases in the error term $e_t$ of (1), together with other high frequency noise.

For an initial cleaning of the underlying time series, three different types of outliers had been considered:

- additive outliers
- level shifts
- transitory components.

Additive outliers appear at one point in time and vanish thereafter without having any lasting effect on the further development of
the series. In a graphical representation of a time series they are marking a one-time spike in the plot\textsuperscript{32}.

Level shifts are innovations which mark a break in the time series, shifting the successive mean by a constant value. They appear as an upward or downward step in the time series.

Transitory components are outliers lasting only for a limited time and vanishing thereafter. They are either fading out in successive time periods (decay) or they evolve slowly over time and vanish suddenly (ramp).

Some of these outliers can be explained by variables outside this model. Ignored working day variables or weather conditions – influencing the economic output – are examples. Furthermore, every unusual strong movement in the history of a time series may appear as an outlier. In this special case here, this goes for sharp recessions, too. It is a difficult task to detect outliers and one has to define first of all what should be considered an outlier and by which value it has to be replaced.

For the present study, outlier detection has been carried out by modelling all time series as seasonal ARIMA models with regression effects (sometimes also called Reg-SARIMA models for short), which took care of the varying number of working days of different countries by regressing on them. Several steps of estimation are carried out in order to check whether an elimination of the significant outliers (in this study dummy variables with a $t$-value higher then 3.8) improves the following model\textsuperscript{33}. A purely me-

\textsuperscript{32} It has to be borne in mind that in a first-order differenced time series an additive outlier marks a spike with a counter-reaction in the next period. If no counter-reaction follows, it appears as a level shift in the original series.

\textsuperscript{33} This procedure was assisted by the software package TRAMO developed by Gomez – Maravall (1992). It is widely used also in business cycle studies like for example in Artis – Krolzig – Toro (2004), Artis – Marcellino – Proietti (2004) and Altissimo et al. (2001).
chanical detection procedure according to some statistical criterion can substantially influence the result of a study. In order to avoid these pitfalls, outliers have been discussed with experts and considered in the light of economic history.

3.2.2 Calendar effects

Apart from the business cycle, the varying number of working days influences economic output, too. Therefore, time series have to be cleaned for this influence. As far as they have not already been derived from the data base in adjusted form, this adjustment was made by the same SARIMA model combined with a regression on working day information as it was used for seasonal adjustment. As not only the varying number of working days constitutes the calendar effect, but also Easter and the leap year effect\textsuperscript{34}, their influence had been considered, too.

Apart from testing for a significant Easter and leap year effect, two specifications for modelling the working day effect have been applied. One tests for the significance of the varying number of working days in total and the other checks for all working days separately. Thus, the regression includes the number of Mondays, Tuesdays, and so on. The discrimination between the two specifications was done according to their \(t\)-values and auto-correlation properties of the residuals from the SARIMA model estimation.

In most applications, as in this study, calendar effects are cleaned from the series under the assumption that they are independent from other components like the trend or the cycle. This orthogonality assumption can be challenged by considerations of possibly larger working day effects in times of high capacity utilisation. During recessions a large number of employees have to work part time and their productivity is probably quite low, so that an addi-

\textsuperscript{34} Other holidays like Christmas have not to be considered separately, as they occur always in the same quarter of the year.
tional working day due to specific calendar constellations will not yield the same value added as in a boom. As a consequence, the business cycle reaction of calendar effects shows up wrongly in the business cycle itself, if calendar effects are estimated as being orthogonal. But this should bias only the amplitude of the business cycle, but not interfere with the dating of the cycle.

3.2.3 Seasonal variations

These high frequency movements can interfere with the dating of the overall peaks and troughs. Sims (1993) and Hansen – Sargent (1993) have shown that seasonal pre-filtering is essential in business cycle analysis in order to avoid a distortion of the business cycle pattern.

Therefore, the largest part of studies is based on the analysis of seasonally adjusted time series, but there are also good arguments for considering unadjusted series as well. Cubadda (1999) found that in the case of existence of seasonal co-integration between analysed time series, the results for unadjusted and adjusted time series can be different. Despite the theoretically difficult economic implication of having a steady state trend showing seasonal variations, he argues in favour of using unadjusted time series and clearing them for seasonal variations within the same framework used for business cycle analysis. In a similar direction points the study by Jäger – Kunst (1990) who showed that the process of seasonal adjustment spuriously amplifies the persistence of variables.

There are several approaches of clearing for seasonal variations which estimate the trend and the cycle within one framework, as proposed by Cubadda (1999). In unobserved components

35 It has to be noted, that the application of band-pass filters for frequencies of the business cycle not only cancels out the trend component but also seasonal variations. It is unclear whether this is conforming to a separation process within one framework in the sense of Cubadda (1999).
models, this is done either by assuming a fixed trigonometric movement of seasonality\textsuperscript{36} or by regressing on dummy variables. In both cases, seasonality is modelled as being stable over the whole time span of the series, which seems to be a rather strong assumption. So the advantage of isolating the seasonal variation within the same framework used for identification of the business cycle has to be weighed against the advantage of allowing for a flexible trend for seasonal variations.

The most popular methods for univariate seasonal extraction are X-12 from the US-Census Bureau and TRAMO-SEATS developed by the Bank of Spain\textsuperscript{37}. Both procedures allow – in contrast to unobserved component models with a basic structure – a variation of the seasonal component over time. This captures the possibility of moving seasonal patterns due to technical progress or changing economic circumstances\textsuperscript{38} which could be very important for cleaning longer time series like the ones used for business cycle analysis.

Weighing the pros and cons of the different approaches, it was decided to separate the seasonal component externally. This was done using the TRAMO-SEATS software. Fiorentini – Planas (2003) have shown that business cycle dating is not sensitive to the use of either X-12 or TRAMO-SEATS. This can be seen in Figure 2.

Nevertheless, both approaches, the unobserved component model and the univariate time series model, assume orthogonality between seasonal variations and the business cycle, which should be borne in mind when interpreting the respective results.

\textsuperscript{36} Artis – Marcellino – Proietti (2004) or Hahn – Walterskirchen (1992) are examples for this approach.

\textsuperscript{37} See Gomez – Maravall (1992).

\textsuperscript{38} Reasons for that could be new processing techniques in the construction sector stabilising output during winter periods or changing seasonal patterns in tourism.
3.2.4 The trend

The deviation cycle approach requires a separation of the business cycle fluctuations from the trend component of a series. There is no definition of the properties that such a trend should possibly have. There is consensus only about the residual part of time series after de-trending. Cyclical variations should be second order stationary, show some autocorrelation characteristics and are expected to have a cycle frequency of some years. According to several authors, this type of cycle is best mirrored in the capacity utilisation of enterprises, which varies auto-regressively without any trend. Therefore, the analysis of business cycles is in most cases based on figures showing these patterns of changing economic activity.

As there is no consensus concerning the shape of the trend, there exists a plethora of approaches to de-trending. Unfortunately, according to Canova (1998), these different methods of de-trending or direct business cycle extraction can give substantially different results for dating the business cycle. Considering this, various methods of de-trending will be used in the present study. The selection of these techniques will be based on their popularity as well as their appropriateness for analysing the Austrian business cycle.

In order to judge on the latter, a theoretical evaluation together with reported empirical features of the different approaches will be presented.

3.2.4.1 Deterministic trend

Due to its simplicity, this approach was historically very popular in business cycle analysis. A linear (or log-linear) trend – in most cases obtained by fitting a regression line – was deducted from the time

39 See e.g. Tichy (1994) or Breuss (1984).
series. The residual is taken to represent the business cycle (and
the remaining components). Instead of a linear trend, other poly-
nomial functions of time are possible. With the study of Nelson –
Plosser (1982), this approach has lost most of its popularity. For the
U.S., they found that a large part of economic time series are al-
legedly difference stationary, where the trend cannot be re-
moved by linear de-trending.

For this reason, this method is not state-of-the-art and should not
be considered as an option further on in the present study.

3.2.4.2 Phase-average trend

This is a heuristic method of determining the trend of non-station-
ary time series, usually applied in the classical business cycle
framework for estimating the amplitudes of business cycle vari-
ations of trended time series. Basically, it consists of the following
steps\(^{40}\): After identifying the turning points in the trended time se-
ries, it is split up into segments between two consecutive turning
points (called "phases"). For each of these segments the mean is
calculated (this is the "phase-average"). In order to obtain the
trend for the whole series, these averages are combined with a
smoothing three-term moving average. As this method of trend
extraction requires an ex-ante determination of the turning points,
it is not suitable for this study\(^{41}\).

3.2.4.3 First-order differencing

Here, de-trending is made by deducting the value at time \(t - 1\)
from \(t\) in order to get stationary differences (if they are first-order
difference stationary which is in most cases assumed or tested). If

\(^{40}\) A detailed description of the full procedure can be found in Boschan – Ebanks
(1978).

\(^{41}\) Breuss (1984) has used this approach for isolating the Austrian business cycle.
this method is applied to a series for which the logarithm has been taken, the outcome can be interpreted as growth rates.

Therefore the basic assumptions behind the method of applying first-order differences are that the trend component of the series corresponds to a random walk process without drift, that the cyclical component is stationary and that the two components are uncorrelated\(^{42}\). The original time series \(y_t\) is assumed to have a unit root, which is entirely due to the trend component of the series\(^{43}\). This assumption of a random walk trend gives anything but a smooth profile of the trend component, which is for example the case in the Solow growth model. Every shock to this trend will never decay in impact in the future. Nevertheless, this assumption is still popular in empirical economics since the pioneering work of Nelson – Plosser (1982) who found unit roots in 13 out of 14 long-term annual US macro series, including real GDP\(^{44}\).

In the case of a time series model based on seasonally adjusted data, this behaviour can be represented as

\[
(2) \quad y_t = x_t + c_t + \varepsilon_t,
\]

\(y_t\) is our trended aggregate, \(c_t\) the cycle to be extracted and \(\varepsilon_t\) some white noise process where \(\varepsilon_t \sim N(0, \sigma^2)\). The assumption of a random walk trend implies that \(x_t = y_{t-1}\) which, entered into (2), gives

---

\(^{42}\) See Canova (1998).

\(^{43}\) There are also multivariate forms of this approach. Cheung – Westermann (1999) estimated the trend of industrial net production as the co-integration relation between Germany and Austria. Business cycle effects are captured by the residual short-run relationship not included in the error correction term.

\(^{44}\) Since then, several authors (e.g. Rudebusch, 1993 and Diebold – Senhadji, 1996) have challenged these findings by showing that the testing procedure has not enough power against economically relevant trend-stationary alternatives.
Simple arithmetics isolates the cycle as

\[(3) \quad y_t = y_{t-1} + c_t + \varepsilon_t\]

It can be seen, that the cyclical component includes the error term, which makes it very erratic and does not show up in the smoothness one would expect from a pure cyclical variation. Furthermore, forming first-order differences does not correspond to a symmetric filter as it generates only differences to past values. This leads to a phase shift of the cyclical component and therefore yields a dating output different from the case of using symmetric filters.

In order to look at the properties of this first-order difference method at the frequency domain, Figure 2 depicts the gain function when filtering data that are already seasonally adjusted. The grey shaded area marks the frequency band of the typical business cycle between \(\pi/3\) (which corresponds to cycles of 6 quarters length) and \(\pi/16\) (corresponding to a length of 32 quarters). The bold red line reveals that the first-order difference filter indeed cancels out frequencies that are located close to zero and therefore can be regarded as variations of the trend component. But it also wipes out mistakenly some spectral mass of the business cycle frequencies in the shaded area. At \(\pi/2\), which corresponds to cycles of 4 quarters) there is no spectral mass at all, as the data have been seasonally adjusted beforehand.

A remarkable property of this filter is that it superimposes all higher frequencies, raising the gain for very high frequencies above 1. This explains the very erratic output of the filter as one can see from looking at growth rate series.

Applying first-order differences to processes which are either trend stationary or show a higher degree of integration biases the outcome for business cycle interpretation. In the first case, this would
correspond to over-differencing which biases the cyclicality to the high frequency spectrum, whereas in the second case under-differencing shifts it to the low frequency area\textsuperscript{45}.

Due to these deficiencies, the first-order difference filter has lost most of his popularity in empirical studies on business cycles. If it is used at all, it is mostly to check for robustness of the output across several methods or to illustrate its deficiencies. Despite these undesired properties which limit the use of this filter for business cycle analysis, it will be considered in this study due to the popularity of growth-rate-based interpretations as a rule-of-thumb method\textsuperscript{46}.

### 3.2.4.4 Hodrick-Prescott filter

The use of the business cycle filter proposed by Hodrick – Prescott (1980) is very popular among business cycle researchers. It is a flexible tool that is capable of removing non-stationary components that are integrated of order four or less\textsuperscript{47}. Effectively, the trend implicitly fitted by the HP filter amounts to a process of curve fitting. It results from constructing a trend as smooth as possible, with penalizing all squared deviations from this trend from the original time series. The precise formula is

\[
\min_{\{g_t\}} \sum_{t=1}^{T} \left[ (y_t - g_t)^2 + \lambda \left[ (g_{t+1} - g_t) - (g_t - g_{t-1}) \right]^2 \right] \quad \lambda \geq 0
\]

where \( y_t \) is the original trended time series and \( g_t \) is the trend to be estimated and subtracted. \( \lambda \) is acting as the signal-to-noise ratio, being the weight for penalizing all deviations from trend, which has to be fixed by the user. If \( \lambda = 0 \), there is no difference between the trend and the original series, if \( \lambda \) approaches infinity.

---

\textsuperscript{45} See e.g. Ritschl – Uebele (2006).

\textsuperscript{46} Like it is done by the CEPR Business Cycle Dating Committee.

\textsuperscript{47} See Baxter – King (1995).
the trend component becomes linear. Typically a value of 1600 is chosen for applications using quarterly data\textsuperscript{48}. The HP filter extracts a trend which can be stochastic but moves smoothly over time and is uncorrelated with the cyclical component\textsuperscript{49}. A further property is its symmetry, so that no phase shift is introduced. In its ideal representation (its infinite sample version), it approximately places zero weight at the zero frequency (the trend) and close to unit weight at high frequency. This last property leads to a non-trended output but which carries high-frequency information like the noise component $e_t$. Figure 2 depicts the gain function of this filter. It wipes out rather clear-cut all frequencies lower than $\pi/16$ (i.e. cycles longer than 32 quarters) and leaves all other frequencies unchanged (again, the loss of spectral mass at $\pi/2$ is due to the preceding seasonal adjustment process).

In order to overcome the disadvantage of residual high frequencies, sometimes a kind of low-term moving average is applied after the HP filter for smoothing the output. Artis – Marcellino – Proietti (2004) used a band-pass version of the HP filter. They combined two HP filters – which both are virtual approximations of high-pass filters – in order to get a smoother output, corresponding to the business cycle frequency band.

The trend resulting from the minimisation process of (5) can also be represented as a linear symmetric filter. Like all symmetric filters, the HP filter is – in its ideal presentation – subjected to the endpoint problem, which means that there is a loss of several observations at the beginning and the end of the series. There exists a trade-off relation concerning symmetric filters, i.e. the more observations one is willing to lose, the more exactly the filtering procedure works. In the case of the practically applied HP filter, there is no such obvious end point problem as there is a kind of built-in me-

\textsuperscript{48} It can be shown that $\lambda = 1600$ corresponds to a cut-off of frequencies lower than 32 quarters. See e.g. Prescott (1986) or Baxter – King (1995).

\textsuperscript{49} See Canova (1998).
chanical forecast feature which allows de-trending even at the endpoints\textsuperscript{50}. The HP filter calculates the trend component and identifies the cyclical component as the difference between the original series and the trend component\textsuperscript{51}. The end-point problem is therefore concentrated on changes in the trend component. This makes the HP filter very attractive for practical purposes, but the end-point problem is solved at the cost of the accuracy of the filter. Especially at the end points, the filter works imprecisely in that it shows a stronger leakage. This means that some frequencies belonging to the trend can pass, whereas fluctuations of cyclical nature (especially those of low frequency order) are filtered out.

Cogley – Nason (1995) and Canova (1998) pointed out that despite the HP filter's ability to de-trend difference stationary time series, it distorts the frequency spectrum so that business cycle extraction could be problematic.

3.2.4.5 The Baxter-King filter

Band-pass filters are understood as frequency filters, which give – in their ideal representation – zero weight to the frequency band to be filtered out and unit weight to the rest\textsuperscript{52}. Baxter – King (1995) proposed such a filter for business cycle filtering, which is an optimal linear approximation to an ideal band-pass filter. They constructed a band-pass filter by starting from a low-pass filter, which allows all frequencies $\omega$ (below or equal a certain threshold $\omega$) to pass. This requires that all frequencies get a unit weight $\beta$ if they are above this threshold and zero otherwise:

\textsuperscript{50} See Baxter – King (1995).
\textsuperscript{51} See Kranendonk – Bonenkamp – Verbruggen (2004).
\textsuperscript{52} In a wider sense, high-pass filters, which filter out all frequencies below a certain frequency threshold (e.g. the trend) as well as low-pass filters, which let pass only frequencies below a certain threshold, can be regarded as band-pass filters. But what is meant here are only the filters which capture the band between two thresholds not belonging to extreme ends.
With the help of these frequency weights \( \beta(\omega) \) it is possible to derive the time series filter weights \( b(h) \) by applying the inverse Fourier transformation to the frequency response function

\[
(7) \quad b_h = \int_{-\pi}^{\pi} \beta(\omega) e^{ih\omega} \, d\omega
\]

The time series filter weights \( b(h) \) can be used for constructing the ideal low-pass filter in the time domain

\[
(8) \quad b_L = \sum_{-\infty}^{\infty} b_h \, L^h
\]

This ideal low-pass filter is symmetrical (as it goes from \( -\infty \) to \( +\infty \)) and is constructed as an infinite-order moving average. The weights in this moving average are \( b_0 = \omega / \pi \), and \( b_h = \sin(h\omega) / h\pi \) for \( h = 1, 2, \ldots \)\(^{53}\). In practice, an approximation of this ideal filter will be enough, such that a shorter filter can be applied which solves the end point problem at the cost of a leakage.

From this low-pass filter, a band-pass filter can be easily derived from two consecutive low-pass filters, one working at the lower boundary \( \omega \) and the other at \( \omega \). The approximation to this ideal band-pass filter with the weighting scheme derived this way is called the Baxter-King filter or BK filter for short. Its weights are given by

\[
(9) \quad \tilde{b}_L(L) = \frac{\sin(L\omega) - \sin(L\bar{\omega})}{L\pi} - \frac{1}{2K+1} \sum_{L=-K}^{L=K} \frac{\sin(L\omega) - \sin(L\bar{\omega})}{L\pi}
\]

\(^{53}\) For details the reader is referred to Baxter-King (1995).
with again \( \omega \) and \( \Omega \) being the lower and upper boundaries and \( K \) representing the length of the filter.

**Figure 2: Gain function of stationary transformations**

The upper and lower boundaries (usually cyclical components between 8 and 32 quarters\(^{54} \)) are captured, by filtering out all frequencies above or below. Convenient properties of this BK filter are that the identification assumption requires no restriction concerning the trend to be either deterministic or stochastic and it al-

---

\(^{54}\) In the frequency domain, this corresponds to a band between \( \pi/16 \) (\( \approx 32 \) quarters) and \( \pi/4 \) (\( \approx 8 \) quarters).

lows for changes in the trend behaviour over time as long as the
changes are not too frequent.\footnote{See Canova (1998).}

An advantage over the HP filter is that the BK filter cancels out
higher frequencies above the cyclical variations, too, whereas the
former – acting as a pure high-pass filter – still gives an erratic but
de-trended output. This can be seen again in Figure 2. The BK filter
cancels out lower frequencies dedicated to the trend frequencies
below $\pi/3$. This allows the filter to be applied theoretically even to
non-seasonally-adjusted series.

A further advantage is that the loss of frequency information due
to data aggregation seems to be less of a problem for band-pass
filters than for the HP filter. Aadland (2005) has shown that aggreg-
gated high-pass filtered data (e.g. by the HP filter) can lead to
spurious cycles at the business cycle frequencies, if the disaggre-
gated data carried strong variations in the high frequency area.
This is due to the so-called "aliasing"-problem which arises when
high frequency data are observed at lower frequencies.

As the BK filter is symmetric like the HP filter, it causes no phase shift.
A convenient feature of the BK filter is its transparency. The user
can explicitly fix the upper and lower level of the band to be fil-
tered; thereby defining what should be understood as the business
cycle. Furthermore, the degree of approximation to the ideal
band-pass version can be chosen. This can be done by sacrificing
observations towards either end point in order to make the filter
work more exactly, i.e. to reduce its leakage. The problem of
leakage arises with the approximation of the filter. As it is not pos-
sible in practice to work with infinitely long time series, shorter filters
have to be applied. This has two consequences: First, frequencies
can pass which should be filtered out and some are mistakenly fil-
tered out which should pass; and second, frequencies are super-
imposed at the borders of the frequency band, which appears as
side-lobes. As Woitek (2001) pointed out, this could possibly lead to spurious results in business cycle analysis. But the leakage problem as well as the one of amplified side-lobes can both be reduced by using longer-term filters and sacrificing more observations at either end of the time series.

Applying the BK filter, one has the possibility to individually select the trade-off between the accuracy of the filter and the end-point problem. Compared with this, the HP filter properties depicted in the gain function of Figure 2 are calculated by accepting a loss of two observations at either end of the time series. It can be seen, that the precision of the filtering process at \( \pi/16 \) is not as exact as in the BK filter case. Furthermore, the part in the middle of the business cycle frequency band is superimposed (values above 1) and there is some leakage at higher frequencies.

Figure 3 shows how the precision of the filtering process changes if one is willing to sacrifice a higher number of observations at the start and the end of the series. Figure 3a shows the gain function obtained by a BK filter with a window length of 6 quarters, causing a loss of 3 quarters at either end. Compared with this, the 12-quarter-window filter (resulting in a loss of 6 quarters at both ends like it is used in our calculations) cuts out more precisely the desired frequencies, without strongly superimposing business cycle frequencies and with a lower leakage. But this precision is obtained at the cost of losing the possibility of analysing the business cycle at the margin. As this is the period for which the dating is most important in order to take timely economic policy measures, this can be regarded as a drawback.

To overcome this problem, some authors propose an extension of the time series by applying forecasting techniques. Fiorentini – Planas (2003) propose an ARIMA forecast for that purpose. But all forecasts based on univariate time series methods (without including external information like business survey data) may reduce the
end-point problem at the cost of risking forecasting errors of possibly the same size\textsuperscript{56}.

### 3.2.4.6 The Christiano-Fitzgerald filter

A somewhat different band-pass filter has been proposed by Christiano – Fitzgerald (2003). It also represents an optimal approximation of an ideal band-pass filter, by imposing a somewhat different criterion of optimality. This criterion minimises the sum of the squared approximation errors which are weighted by their spectral density \( f_x(\omega) \) of the data being filtered\textsuperscript{57}.

\[
(10) \quad \min_{\hat{B}P, \hat{J} = J} \int_{-\pi}^{\pi} \left| B(e^{-i\omega}) - \hat{B}_P(f(e^{-i\omega})) \right|^2 f_x(\omega) d\omega
\]

Furthermore, the filter length is allowed to vary over the time series and is not restricted to being symmetrical. Moving towards the start and the end of the time series the filter becomes more and more asymmetric, which allows circumventing the end-point problem.

Christiano – Fitzgerald (2003) have shown that the length of the BK filter window not only determines the degree of approximation to the ideal band-pass filter, but influences especially its capability of filtering out long-term movements. The filter proposed by them can also be adjusted to filter a specific frequency band, but – in contrast to the BK filter – its length is a result of the optimisation process and cannot be altered by the applier. Therefore, it is not possible to define explicitly the business cycle with regard to a certain frequency band.

\textsuperscript{56} In order to extend the HP filter, Kaiser – Maravall (1999) proposed an IMA (2.2) time series model for forecasting.

\textsuperscript{57} The weights are thus generated by a trigonometric function.
Empirical studies of economic time series for the Netherlands carried out by Kranendonk – Bonenkamp – Verbruggen (2004) have shown a better performance with regard to revisions of the endpoint output than with the BK filter. Nevertheless, the use of asymmetrical type filters suffers from phase shifts or they contain an implicit forecast (based on past observations), respectively. Whether a thorough forecast combined with the BK filter yields a better performance depends on the specific case.

In the literature, modifications of this approach can be found, like in Altissimo et al. (2001) who applied a multivariate Christiano-Fitzgerald filter in order to study business cycles for the euro area. Goldrian – Lehne (1999) proposed a further band-pass filter that is not based on a minimisation process, but on the common pattern of weight matrices, instead.

### 3.2.4.7 The Beveridge-Nelson decomposition

Instead of filtering time series for special frequency bands, Beveridge – Nelson (1981) proposed a time series model-based approach. The idea behind is that after fitting a time series model to the underlying data, its structure can be explored in order to separate the trend from the cycle. The fitting of an ARIMA model implicitly models the trend as a stochastic process. The stationary ARMA part is assumed to be (or at least contains) the business cycle.

In order to show some interesting properties of this approach, a simple example is set up. Assuming the underlying time series can be modelled by an ARIMA (0, 1, 1) process, the time series becomes stationary after forming first order differences. This can be

---

58 In the case of not-seasonally adjusted data, a S-ARIMA model has to be applied.

59 In most cases it will be of order one, which implicitly yields a random walk process.
represented by a moving average process of order 1 and shows at the same time the cycle:

\[ \Delta y_t = e_t + \beta e_{t-1} \]

where \( \Delta y_t \) is the (log) differenced time series, \( e_t \) a white noise term and \( \beta \) represents the moving average parameter with \( \beta < 1 \).

If this expression is solved recursively and the start values are assumed \( y_0 = e_0 = 0 \), the following expression emerges

\[ y_t = \sum_{i=1}^{t} e_i + \beta \sum_{i=1}^{t-1} e_i \]

or

\[ y_t = (1 + \beta) \sum_{i=1}^{t} e_i - \beta e_t \]

In (13) the first term \( y_t = (1 + \beta) \sum e_i \) is the trend part which – being a random walk process – is the sum of its past shocks, and \( \beta e_t \) represents our stationary cyclical part. Equation (13) implies one interesting feature of this kind of splitting the trend from the cycle: The secular as well as the cyclical component are both driven by the same shock at the particular point in time. This means, trend and cycle are perfectly correlated. In order to make this somewhat clearer, we can transform (13) to

\[ y_t = \left[ (1 + \beta) \sum_{i=2}^{t} e_{i-1} + (1 + \beta) e_t \right] + \beta e_t \]

showing that the trend as well as the cyclical component at time \( t \) both depend on shock \( e \). This implication is in stark contrast to the

\[ \text{See Canova (1998), p. 481.} \]
usual assumption of orthogonality of trend and cycle. But it does not necessarily represent a deficiency, as it seems plausible sometimes to see both components influenced by one shock. Admittedly, this means – as criticised by Blanchard – Quah (1989) – lumping together supply side shocks, which affect the secular component, and demand-related shocks influencing the cycle.

As the Beveridge-Nelson approach is based on ARIMA time series models, all problems linked to that sort of modelling are carried over to this trend-cycle separation approach. Foremost, applying this kind of modelling the problem is usually not to find a suitable model but to discriminate between the large amounts of feasible ones\textsuperscript{61}. The identification of the cycle therefore remains a somewhat arbitrary task and may be challenged in academic discussions. Furthermore, ARIMA models became prominent on account of their good short-run forecasting properties, whereas they perform quite poorly for longer-term projections. As a consequence, one can extract quite different trends and cyclical components from a plethora of suitable models.

### 3.2.4.8 Unobserved components models

This type of approach intends to model all components explicitly. A special time series structure is modelled for the trend, the cycle, some regression effects like working days, (possibly the season) and even innovations like structural breaks or suspected outliers\textsuperscript{62}. This method is thus very flexible and a wide variety of specifications is possible. Usually the trend is modelled as some kind of random walk process, possibly allowing for a drift. A prominent representative of this approach is Harvey (1989), who proposed a so-

\textsuperscript{61} Christiano – Eichenbaum (1992) mentioned this problem for practical business cycle analysis.

\textsuperscript{62} For instance, Carvalho – Harvey (2004) estimated a model for the euro area with a trend evolving stochastically in its slope and level with adding a serially correlated stationary component representing the cycle.
called local linear trend where the level as well as the slope is represented by stochastic processes. Depending on restrictions of the variance of these shocks, deterministic as well as random walks (with or without drift) and smoothly evolving trends can result. This flexibility makes it difficult to judge the appropriateness of unobserved components models for business cycle analysis. In some more simple specifications they come close to the Beveridge-Nelson decomposition method.

Morley – Nelson – Zivot (2002) argue that in the unobserved components framework the common (but not necessary) restriction that trend and cycle are uncorrelated leads to great differences in the cyclical output. Whereas the Beveridge-Nelson decomposition implies that a stochastic trend accounts for most of the variation in output, the cyclical variation is dominant in unobserved components models for their part. Lifting the restriction of trend and cycle being uncorrelated, the two approaches can yield identical decompositions. As Blanchard – Quah (1989) assigned supply shocks to the trend and demand shocks to the cycle, the assumption of uncorrelatedness in isolating the cycle can have decisively different consequences for economic policy.

A possible advantage of modelling explicitly all components – as it is done in the unobserved components approach – is that this process can bear closer relation to economic theory than mechanical filtering procedures do. Whereas the purely statistical non-parametric procedures of detrending can be accused of practicing "measurement without theory", Canova (1998) accuses all economic-theory-based decompositions as, "at best, attempts to approximate unknown features of a series and therefore subject to specification errors". So, the explicit modelling strategy not only allows capturing theoretical aspects, but also contains several subjective assumptions.

---

Koopman et al. (2000) give an overview over how certain combinations of these restrictions affect the trend.
Figure 3a: Gain function of a BK filter with a length of 6 quarters

Figure 3b: Gain function of a BK filter with a length of 12 quarters
Furthermore, in practical work it sometimes turns out that the results obtained are not substantially different from less sophisticated approaches. For instance, Hahn – Walterskirchen (1992) pointed out that if the size of amplitudes is measured by simple variation coefficients of growth rates rather than through calculations by their model, results are largely the same.