5. Analysing cyclical comovements

After extracting the business cycle, an analysis about the leading and lagging properties of several economic time series is of great interest. There exists a broad range of statistics in this respect, which can be classified according to their belonging to time or frequency domain statistics. It is worth mentioning that this kind of analysis already forms an implicit part of the dynamic common component approach and constitutes an essential part of the identification and extraction of the business cycle.

5.1 Time domain statistics for analysing comovements

To this class all sort of statistics belong, which exhibit comovements between time series at discrete points in time or for certain intervals. They are mainly based on correlation statistics referring to second moments of time series.

Calculating cross-correlations in order to detect potential leads and lags is the simplest form of identifying comovements between the business cycle and other series. They are constructed as

\[ \rho_{b,s}(\tau) = \frac{\text{Cov}(z_{b,t}, z_{s,t-\tau})}{\sqrt{\text{Var}(z_{b,t}) \text{Var}(z_{s,t})}} \]

with the subscript \( b \) denoting the business-cycle-carrying time series (i.e. the reference series). \( s = 1, \ldots, N \) counts for all other \( N \) time series and \( \tau \) determines the different leads and lags for which cross-correlations are calculated. If \( \tau = 0 \) the synchronised comovement is observed. The classification as leading or lagging is mainly based on the delay with which the highest cross-correlation is achieved. It can only be regarded as clear-cut, if the cross-
correlation is significantly higher than for other leads and lags in close neighbourhood.

**5.2 Frequency domain statistics for analysing comovements**

Sometimes it is easier to interpret comovements not in the time domain, with discrete time data points, but in the frequency domain. This is especially the case in business cycle analysis where special frequencies or frequency bands are the focus of interest. These statistics can be derived by the Fourier transformation of time domain statistics. So is the cross-spectra the frequency-domain-equivalent of the cross correlation \( \rho_{b,i}(\tau) \), like the spectral density function is the equivalent of the auto-correlation function. The cross-spectra of two series for a certain frequency \( \omega \) is defined as

\[
\gamma_{b,s}(\omega) = \frac{1}{2\pi} \sum_{\tau = -\infty}^{\infty} \rho_{b,s}(\tau) e^{-i\omega \tau}
\]

with \( \omega \) being a frequency within \([-\pi, \pi]\) and \( \rho_{b,s} \) being the cross-correlation as defined in (22). As can be seen in (23), the cross-spectrum contains a complex part which does not cancel out because the cross-correlation function is not symmetric, i.e. \( \rho_{b,s}(\tau) \neq \rho_{b,s}(-\tau) \). Therefore, this statistic cannot be interpreted in order to determine leads and lags directly, but has to be transformed into another statistic like the coherence, the phase spectra or the mean delay.
5.2.1 Coherence

The coherence measures the linear relatedness of two stationary processes. It can be regarded as the frequency-domain-equivalent to the cross-correlation in the time domain. The output is defined over the interval $[0, \pi]$ and shows the correlation of the cyclical (or at least stationary) component of the series at each frequency. It is defined by the squared cross-spectrum, divided by the product of the spectral density functions of both series

$$co(\omega) = \frac{|\gamma_{b,s}(\omega)|^2}{\gamma_{b,b}(\omega) \gamma_{s,s}(\omega)}$$

Applying this quadratic transformation ensures that values are real and symmetric. It can be interpreted as the frequency-domain-counterpart to $R^2$, the well known coefficient of determination, as it shows the proportion of variance of one series explained by the other for a given frequency $\omega$.

But this transformation has a substantial disadvantage. Croux – Forni – Reichlin (1999) stressed that this statistic "does not measure correlation at different frequencies, because it disregards the phase difference between variables". Thus, only the synchronised comovement of two time series, over some specified spectrum or at a certain frequency, can be observed. Whether these frequencies are phase-aligned or not is of no influence to this measure. Related to our problem, the coherence assumes high values if both series show similar frequency gains, irrespective of whether one time series is leading or lagging.

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81 In the literature, this measure is sometimes called squared coherence.

5.2.2 Phase spectra and mean delay

In order to supplement the coherence with a statistic that informs about leading and lagging properties of time series, the phase spectrum can be calculated. This can be done by combining the real and the complex part of the cross-spectrum as defined in (23) and calculating the arcustangens of it in order to obtain the phase angle:

\[
\begin{align*}
\gamma_{b,s}(\omega) &= \frac{1}{2\pi} \sum_{r=-\infty}^{\infty} \rho_{b,s}(\tau) e^{-i\omega r} = \left| \frac{1}{2\pi} \sum_{r=-\infty}^{\infty} \rho_{b,s}(\tau) \right| e^{-iPH(\omega) r} \\
\text{where } PH(\omega) &= \text{the phase-delay-generating function multiplied by a scalar, and further} \\
\xi(\omega) &= \arctan \left( \frac{1}{2\pi} \sum_{r=-\infty}^{\infty} \frac{\rho_{b,s}(\tau)}{PH(\omega) r} \right)
\end{align*}
\]

\(\xi(\omega)\) is the phase spectrum over all frequencies, indicating how large the lead (positive numbers) or lag (negative) is. Averaging over a specified frequency band yields the mean delay for this term. In this way one can look at the leading and lagging properties only for frequencies within the business cycle boundaries or for other frequencies in the focus of interest.

5.2.3 Dynamic correlation

Another way of overcoming the deficiency of the coherence of not accounting for phase shifts of frequencies has been proposed by Croux – Forni – Reichlin (1999). They recommended a measure that looks quite similar to its time-domain-equivalent, the cross-correlation, shown in (22):

\[
\tilde{\rho}_{b,s}(\omega) = \frac{\gamma_{b,s}(\omega)}{\sqrt{\gamma_{b,b}(\omega)\gamma_{s,s}(\omega)}}
\]
but this time defined only for $\omega$ within the interval $[0, \pi)$. This measure is related to coherence – as it shows its real part – but instead of rendering values real by a quadratic transformation, the complex term is cancelled out by summing negative and positive waves of the same frequency. This can be seen in a different representation of

$$\tilde{\rho}_{b,s} (\omega) = \frac{\gamma_{b,s} (\omega) + \gamma_{b,s} (-\omega)}{2}$$

This should not be a drawback as $(\omega)$ and $(-\omega)$ have the same periodicity, which is the focus of interest in business cycle analysis. Furthermore, the authors stressed that dynamic correlation observed at a special frequency band is theoretically fully equivalent to static correlation applied to band-pass pre-filtered data accordingly.

5.2.4 Cohesion

In order to analyse the comovement of more than one time series, Croux – Forni – Reichlin (1999) proposed a multivariate version of the dynamic correlation index. It is constructed by weighing together all dynamic correlation coefficients as defined in (27) and is called cohesion by them

$$\text{coh}_b (\omega) = \frac{\sum_{i \neq j} w_i w_j \tilde{\rho}_{i,j} (\omega)}{\sum_{i \neq j} w_i w_j}$$

the $\rho_{i,j}$ being again the dynamic correlation coefficients as defined in (27) over all combinations of time series except their diagonal elements. The authors suggest choosing the weights according to their economic significance (e.g. by their proportion of contributing to GDP). This statistic has the advantage of allowing to capture common comovements of the underlying set of time series at all frequencies at one glance. This makes it more appropriate for certain tasks than looking at cross-correlations because
high values of the latter neither imply nor are implied by co-integration, common cycles or common features\textsuperscript{83}. A practical example of application of this statistic can be found in Croux – Forni – Reichlin (1999) and in Partridge – Rickman (2005).