Chapter 5

The Dynamics of Market Shares

5.1 Existence, Multiplicity, and Stability of Equilibria

Economists traditionally assume that economic systems are in equilibrium. If a system were not, the argument goes, then some individuals do not optimize, or have wrong beliefs. In the long run, such a state cannot persist. Therefore, at least in the long run, a system must necessarily be in equilibrium.

But given that a system might initially be out of equilibrium, how does it get to equilibrium? Usually economists refer to some kind of evolutionary pressures or learning processes which drive the system to equilibrium. However, this answer raises new questions. First of all, it is well known that some systems do not even have an equilibrium. As an example, recall Proposition 1 above. It can be shown that for high substitutability, i.e. large values of $\sigma$, a network equilibrium (in pure strategies) does not exist. In such a case, economists have difficulties to say anything about the outcome of competition.

Another problem is the possible multiplicity of equilibria. It is quite common that economic systems admit multiple equilibria. As an example, we have already discussed the potential multiplicity of consumer equilibria in Section 2.3.1. If several equilibria exist, the question arises, which of the equilibria will be the long run state of the system. As we have argued previously, some
equilibria turn out to be dynamically unstable. Such equilibria, like the shared market consumer equilibrium in the three equilibria case cannot be long run outcomes. However, it is quite possible that there are multiple stable equilibria. Which equilibrium prevails in the long run is then often dependent on initial conditions. Under such circumstances we say the system exhibits path dependence.

The situation seems most predictable if there is a unique equilibrium, as is the case for the network equilibrium we analyzed in the previous chapters under certain conditions. Nevertheless, as soon as we leave the simple case of a one-dimensional state space, even a unique equilibrium need not suffice to predict long run behavior of a system. First, a unique equilibrium might well be unstable. Second, even asymptotic stability does not guarantee a predictable long run outcome. What we need is global asymptotic stability. By definition, global asymptotic stability guarantees that from every initial condition the system converges to the equilibrium in the long run. However, global asymptotic stability is a rather special case if the state space is two- or higher-dimensional.

5.2 Out-of-Equilibrium Dynamics

5.2.1 Dynamical Systems

Stability is not a property that equilibria do or do not possess by themselves. When speaking of stability of equilibria, we must refer to some underlying dynamics. Which dynamics to assume is always a matter of the special situation we are dealing with. Dynamical systems can be set up in discrete or continuous time, they can be stochastic or deterministic, they may be autonomous (time-independent) or nonautonomous, they can admit unique or multiple solutions, and so on.

In this work we consider only deterministic systems, i.e. dynamical systems which involve no stochastic elements. Such systems are usually easier to work with. If a deterministic system has unique solutions, its behavior is in principle completely predictable. Given any initial state, we can compute the state for any later point of time. However, for practical purposes this is often not sufficient. Such systems might well suffer from sensitivity to initial conditions, which means that arbitrarily close initial states diverge
after relatively short amounts of time. This leads to chaotic behavior, making these systems practically unpredictable for longer horizons.

When dealing with dynamical systems which describe the behavior of an economic system, we must distinguish between Nash equilibria of the economic system and dynamic equilibria of the dynamical system. The latter are usually called stationary points of the dynamical system, and we stick to this term in order to avoid confusion.

What are the forces determining the behavior of dynamical systems? When these systems are set up to model the behavior of a group of economic agents, it is usually the strategic incentives which drive the behavior of agents, and therefore the dynamical system. By definition, Nash equilibria are states in which no agent has an incentive to deviate from his strategy. Hence Nash equilibria typically are stationary states of the dynamics. However, in a non-strict equilibrium some agents have no incentive not to deviate, either. So, at least if there are multiple solutions, such equilibria need not be stationary states. On the other hand, stationary states need not be equilibria for some dynamics. To see this, imagine a dynamics based only on imitation of successful agents. If we start with a homogeneous population, i.e. with all agents using the same strategy, then there is nothing else to imitate, and the system will necessarily remain in this state, even if it is not an equilibrium.

5.2.2 Levels of Rationality

Dynamical systems describing the behavior of a population of agents can roughly be classified along a line indicating the level of rationality they assume. At the zero-rationality end of this line are the pure evolutionary dynamics, the best-known of which is the replicator dynamics (see Maynard Smith, 1982). This dynamics assumes that agents are hard-wired to some strategy throughout their life, and that this strategy is inherited by their offspring. Payoffs are interpreted as biological fitness in this model. The more successful a strategy, i.e. the higher an individual’s payoff compared to the average payoff in the population, the more offspring this individual has. Hence the frequency of relatively successful strategies increases through a purely evolutionary force. This dynamics stems from the biological sciences and its usefulness in economic contexts is at least questionable. Interestingly

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1For an overview of imitation learning see Schlag (1998).
however, it has been shown (Börgers and Sarin, 1997) that this dynamics also arises from a model of agents adapting by reinforcement learning.

On the other end of the line there are the dynamics which assume full rationality of agents, and sometimes even more than that. For example, Matsui and Matsuyama’s (1995) perfect foresight dynamics assumes that agents are ‘omniscient’, implying that they have perfect foresight and know from the outset which path the dynamical system will take. Given this knowledge, they optimize their selection of strategies along this path, and as a result, in aggregate the system follows exactly the path they have foreseen.

Between these two extremes, there is the large area of bounded rationality. The dynamics in this area have in common that they assume agents are boundedly rational, groping for optimality under the constraints of limited computing power, limited foresight, and/or limited knowledge and information about the behavior of other agents. This area includes a lot of rather different models of behavior which have been developed during the last decades, e.g. models of reinforcement learning, imitation learning, best response dynamics, fictitious play, etc. Some of these dynamics have been designed for small groups of players, others for large interacting populations.

Our topic of interest is the behavior of consumers and network operators in a telecommunications market. In order to point out the difficulties arising in a dynamic analysis of this market we concentrate on consumers’ behavior for a given price structure of the networks. Hence we assume that networks’ prices and fixed fees are constant, and consumers adapt their choice of network subscription to the changing market shares induced by these choices. What is the appropriate dynamical system to set up for such a model? To answer this question we make several simplifying assumptions, but try to stay close to what we consider consumers’ real world behavior to be like.

### 5.2.3 Simplifying Assumptions

First of all, given that the number of individuals participating in mobile telephony is several millions even in smaller countries, we assume that we are dealing with an infinite population of consumers. We also assume that all these consumers are identical, they do not differ in their demand for phone calls. They also do not prefer any of the networks per se, so in contrast to
the previous chapters we assume here that networks are not differentiated from the consumers' perspective.

We further simplify the analysis by neglecting the obstacles which consumers usually encounter when trying to change their network subscription. The sum of these obstacles, be they of monetary nature or not, expressed in monetary units, is called *switching costs*. We simply assume that there are no switching costs. While this assumption might be called unrealistic, it does not play an important role in our analysis. Moreover, with number portability (the possibility to keep one's phone number when changing the network) likely to be implemented in many mobile telecommunications markets in the near future, actual switching costs will be greatly diminished.

As in the previous models, we stick to the assumptions of a covered market and of balanced calling patterns. As already discussed in the introduction, this implies that the percentage of calls a consumer makes into a certain network is exactly equal to the market share of this network.

Consumers cannot switch whenever they like. This reflects the fact that subscription contracts, which usually include a heavily subsidized handset, are binding for some period of time, most commonly 12 months. This induces *inertia* in the dynamics. Not all consumers get the possibility to switch at the same time. Instead, every time period only a small fraction of consumers is allowed to switch. For our purposes the most natural time period is one day. Given that subscription decisions are binding for 12 months, and assuming that contract initial dates are uniformly distributed over the days of the year, each day about one in 365 consumers is allowed to switch. However, for convenience we assume that each day the consumers getting a switching possibility are randomly selected from the population. While this implies that the same consumer may be allowed to switch on two consecutive days, it makes the analysis tractable by freeing us from the need to keep track of the contract length of each individual in the population. Furthermore, a consumer's *expected* waiting time till the next switching possibility is still 365 days, and we will see that this assumption does not change the direction of movement of the state space but only the velocity, implying that the orbits of the corresponding dynamical systems are the same.

Concerning consumers' beliefs, we assume that consumers have full information about current market shares. Thus, consumers' beliefs are correct, they know networks' current market shares at any point in time. This as-
assumption is a strong one, but it may be justified by consumers’ possibility to inform themselves about current market shares from the media, or by consumers’ ability to estimate these market shares from the distribution of network choice among their calling partners.

However, we assume that consumers are myopic, i.e. shortsighted. This means they act as if they believe that during the time period where their contract is binding the market share will not change. This assumption appears not too unrealistic, since it is often observed that consumers act shortsighted in their daily decisions, particularly if the stakes are not very high. Furthermore, we assume that consumers are rational. Given their beliefs, consumers optimize, i.e. they make an optimal subscription decision given the current market shares.

In the absence of termination-based price discrimination market shares do not matter for a consumer’s subscription decision. Indeed, in this case for generic combinations of price and fixed fee there is always a network corresponding to a strictly dominant strategy. For any given initial market shares, this dominant network’s share will monotonically increase to 1.

The interesting case is the one with termination-based price discrimination. If on-net prices differ from off-net prices, it depends on the market shares, which network is currently optimal. While the state of the population moves to this ‘corner’ of the market, the optimal network may change, inducing the population state to change the direction of movement, and the question then is, what behavior this dynamical process will show in the long run.

Note that the system as we have set it up need not admit unique paths of the state. If for some distribution of market shares there are several optimal networks, we do not prescribe a certain choice to the consumers. In such a case, it is possible that some fraction of switching consumers chooses one of the optimal networks, while the remaining fraction chooses another one of those. Starting from such a population state, several different future paths are possible.

The system as described up to now is in discrete time. Given that the stepsize of the system, i.e. the distance between successive states, is rather small (at most \( \frac{1}{365} \) of the distance between two cornered market outcomes in the state space), we will approximate this discrete system by one in continuous
time. This facilitates the analysis and does not change the system’s behavior qualitatively.

5.3 Best Response Dynamics

Let us now state the dynamic model of this chapter in a more formal way. There are $n$ telecommunications networks. Each network $i$ charges its customers a price of $p_{ij}$ for a single call into network $j = 1, \ldots, n$. We denote by $p_{ii}$ network $i$’s on-net price and by $p_{ij}$ for $i \neq j$ network $i$’s off-net prices.

There is a continuum of agents, each of whom can choose among $n$ networks to subscribe to. Each agent makes telephone calls to other agents. Demand for calls is completely inelastic, and we assume that each agent calls a fixed number, normalized to 1, of randomly selected other agents during a unit time period. Agents’ total (active and passive) utility from a call is $U$. Utility is quasilinear in money, i.e. an agent’s net surplus, given market shares $x = (x_i)_{i=1, \ldots, n}$ with $\sum_i x_i = 1$, is

$$v_i(x) = U - \sum_{j=1}^{n} p_{ij} x_j,$$

if the agent is subscribed to network $i$. Here we assume that $U$ is large enough to prevent net surplus from becoming negative.

Time $t \geq 0$ is continuous, and in each small time interval $dt$, a randomly selected fraction $\delta dt$ of agents receives a switching opportunity. An agent receiving a switching opportunity chooses a myopic pure best response from the set $BR(x(t))$ of (pure or mixed) best responses to $x(t)$, i.e. he subscribes to a network $i$ maximizing his instant expected payoff $v_i(x(t))$.

With these assumptions agents are playing a population game. Networks correspond to pure strategies, and the vector $x$ of market shares is the state of the population, corresponding to the population’s mixed strategy. Each agent repeatedly plays a symmetric two-person $n \times n$ game (a matrix game) against his calling partners. Since the latter are randomly drawn from the population, and since agents are (myopic) expected utility maximizers, we
can say each agent plays ‘against the population’. The payoff matrix of the
game is given by the \( n \times n \) matrix

\[
A = \begin{bmatrix}
U - p_{11} & \cdots & U - p_{1n} \\
\vdots & \ddots & \vdots \\
U - p_{n1} & \cdots & U - p_{nn}
\end{bmatrix}.
\]

The motion of the population state \( x(t) \in S_n \) is then described by the set of differential inclusions \( \dot{x}(t) \in \delta[BR(x(t)) - x(t)] \). Normalizing \( \delta = 1 \), we finally obtain

\[
\dot{x} \in BR(x) - x.
\] (5.2)

This dynamics is known as the best response dynamics. It was originally formulated by Gilboa and Matsui (1991) and Matsui (1992). Mathematically it is equivalent to Brown’s (1951) continuous-time fictitious play process with identical initial moves. These dynamics has thoroughly been studied by Hofbauer (1995), see also Berger (2001, 2005), and we will utilize their results for our analysis.

The right-hand side of (5.2) is set-valued, since best responses need not be unique. Therefore (5.2) is a system of differential inclusions rather than a system of differential equations. For details on differential inclusions see e.g. Aubin and Cellina (1984). The right-hand side of (5.2) is upper-semicontinuous with closed and convex values, guaranteeing existence of solutions through any initial value. However, in general these solutions need not be unique.

Obviously, if the pure strategy \( i \) is the unique best response to \( x(t) \), the best response path through \( x(t) \) is a straight line, heading for \( i \), as long as this strategy remains the unique best response. The sets of states \( x \) with \( BR(x) = \{i\} \) for different pure strategies \( i \) are disjoint, open and convex subsets\(^2\) of \( S_n \). If after some time the best response changes to \( i' \), then the path suddenly heads for strategy \( i' \). At the turning point \( x \) the respective payoffs are equal: \( u_i(x) = u_{i'}(x) \).

If there is a constant solution through some point \( x^* \in S_n \), then we must have \( 0 \in BR(x^*) - x^* \), or \( x^* \in BR(x^*) \), which means that \( x^* \) is a (symmetric) Nash equilibrium of the game \( A \).

\(^2\)Some of these sets might be empty. This is e.g. the case, if some pure strategy \( i \) is strictly dominated.
As already mentioned, revising agents choose some pure strategy which is a best response to the current state $x(t)$. If there is more than one such pure best response, then the target of the best response path can be any convex combination of these pure best responses. In fact, $\dot{x}(t)$ is not uniquely determined for such a point in time. A consequence of this is that non-strict Nash equilibria need not be stable. While there is always a constant solution through them, there may be other solutions leaving such an equilibrium.

5.4 Two Networks

Let us briefly review the dynamics of market shares in the standard case of two competing networks. Generically there are 4 different cases:

- Network 1 dominates network 2.

  In this case $U - p_{11} > U - p_{21}$ and $U - p_{12} > U - p_{22}$. Put simply, $p_{1j} < p_{2j}$ for $j = 1, 2$. Then network 1 is always the unique best response. No matter what the market shares, each consumer subscribes to network 1 whenever he gets a revision opportunity. The unit vector $e_1 = (1, 0)^t$, corresponding to the state where all consumers are subscribed to network 1, is the unique Nash equilibrium of the respective matrix game and the unique globally asymptotically stable stationary state of (5.2). Interestingly, however, if the process starts out of equilibrium, the equilibrium is not reached in finite time. If $x(t)$ denotes the market share of network 1, then (5.2) reads $\dot{x}(t) = 1 - x(t)$, with solution $x(t) = 1 - (1 - x(0))e^{-t}$. Thus, if $x(0) < 1$, $x(t) < 1$ for all $t$. The same is generally true whenever the best response path converges to a pure Nash equilibrium. However, this is an artefact of our assumptions of a continuum of consumers and of revision opportunities arriving randomly. It is not important for our main arguments.

- Network 2 dominates network 1.

  This is just the mirror image of the first case. $x = 0$ is the unique Nash equilibrium and is globally asymptotically stable.

- Both networks can corner the market.
For this case both $x = 0$ and $x = 1$ must constitute Nash equilibria. This is the case if $p_{11} < p_{21}$ and $p_{22} < p_{12}$, i.e. each network’s on-net price is below the other network’s off-net price. Note, however, that in this case there is always a third equilibrium $0 < \bar{x} < 1$ in mixed strategies, where a fraction $\bar{x}$ of the population subscribes to network 1 and the remaining fraction $1 - \bar{x}$ to network 2. In this equilibrium, all consumers must be indifferent between the networks, i.e. $p_{11}\bar{x} + p_{12}(1 - \bar{x}) = p_{21}\bar{x} + p_{22}(1 - \bar{x})$, from which network 1’s market share can be calculated as

$$\bar{x} = \frac{p_{22} - p_{12}}{p_{11} - p_{21} + p_{22} - p_{12}}. \quad (5.3)$$

For market shares of network 1 below this value, network 2 is the best response, and if the market share of network 1 is higher than $\bar{x}$, consumers prefer network 1. As a consequence, any best response path starting at $0 \leq x(0) < \bar{x}$ converges to 0 and any path starting at $\bar{x} < x(0) \leq 1$ converges to 1. In the knife-edge case $x(0) = \bar{x}$ there are infinitely many solutions: A path can remain at $\bar{x}$ for an arbitrary amount of time and then head off to either of the two pure equilibria. From this analysis, both pure equilibria are locally asymptotically stable, while the mixed equilibrium is unstable.

- None of the networks can corner the market.

In this case there are no pure equilibria, i.e. $p_{ii} > p_{ji}$ for $j \neq i$. Since the on-net price of a network is above the rival’s off-net price, best response paths point inwards at both boundary points $x = 0$ and $x = 1$. Again there exists a unique equilibrium in mixed strategies given by (5.3). However, this time the mixed equilibrium is globally asymptotically stable. This corresponds to the case of the stable shared market equilibrium we analyzed in the previous chapters. Any deviation of network 1’s market share to a value below $\bar{x}$ makes all consumers prefer to subscribe to network 1. Hence $x(t)$ rises until $x(t) = \bar{x}$ again. Analogously, if a deviation to a value above $\bar{x}$ occurs, $x(t)$ falls until equilibrium is re-established, and then remains there.

The four generic scenarios described above are valid not only for the best response dynamics, but for virtually any dynamics that respects the basic
assumption that the driving force is consumers' incentives. More precisely, any payoff monotonic dynamics, i.e. any dynamics with the property that 
\[ \dot{x}(t) > 0 \text{ if } v_1(x(t)) > v_2(x(t)) \] and vice versa, shows the same qualitative behavior as the best response dynamics in the case of two networks. The reason for this is that the state space is one-dimensional, and the direction of movement is completely determined by the payoff differences the consumers face. This is no longer true if we move to higher-dimensional state spaces. In the next section we treat the simplest of these non-trivial cases, the case of three networks.

5.5 Three Networks

If we have three different networks in the market, i.e. \( n = 3 \), then the state space is the two-dimensional probability simplex \( S_3 = \{ x = (x_1, x_2, x_3)^t : x_i \geq 0, \sum x_i = 1 \} \). If a best response path \( x(t) \) changes direction, then at the turning point consumers must be indifferent between two of the three networks, and they must (weakly) prefer these two to the third one. Denote these two networks by \( i \) and \( j \), and the third one by \( k \), then the indifference condition is \( v_i(x) = v_j(x) \geq v_k(x) \), or \( (Ax)_i = (Ax)_j \geq (Ax)_k \). The equality defines an affine linear subspace in \( S_3 \), which is generically a hyperplane, i.e. a line. The weak inequality then determines a (possibly empty) halfline in \( S_3 \) which we denote by \( li_{ij} \). If the three indifference halflines meet in a point \( x^* \) in \( S_3 \), then this state is a Nash equilibrium (a completely mixed equilibrium, a partially mixed equilibrium, or a pure equilibrium, depending on whether it is in the interior of \( S_3 \), in the interior of one of the faces where \( x_i = 0 \) for some \( i \), or at a vertex). If the indifference lines do not intersect in the interior of \( S_3 \), then all Nash equilibria are on the boundary of the simplex.

We start our analysis by picking out three classes of price structures having particular symmetry properties. This allows us to focus on different important types of dynamic behavior illustrating our main points.

5.5.1 On-net Price below Off-net Price

Consider the class of games with the property that \( p_{ii} = p \) and \( p_{ij} = q \) for all \( i, j \in \{1, 2, 3\} \) and \( i \neq j \). This means that all three networks charge the same
on-net price $p$, and the same off-net price $q$ for calls to either of the other two networks. Assume furthermore that $p < q$, i.e. it is cheaper to call on-net. A consequence of this is that each network can corner the market. To see this, assume network $i$'s market share $x_i$ is close enough to 1. Then every consumer prefers to subscribe to this network, since almost all his calls terminate in network $i$. The situation is qualitatively equivalent to the analogous case with only two networks. All three networks constitute strict Nash equilibria. As in the two-networks case, there is also a completely mixed equilibrium $\bar{x}$, which by symmetry of the networks lies in the barycenter of the simplex, $\bar{x} = (1/3, 1/3, 1/3)^t$. In addition to this completely mixed equilibrium there are three partially mixed equilibria on the three boundary faces of the simplex where one of the $x_i$ vanishes. These three boundary faces just correspond to the one-dimensional simplex serving as the state space in the two-networks case above. By symmetry, the three partially mixed equilibria are

$$(1/2, 1/2, 0)^t, \quad (1/2, 0, 1/2)^t, \quad (0, 1/2, 1/2)^t.$$ (5.4)

To see that these points are indeed equilibria, consider as an example the state $(1/2, 1/2, 0)^t$. Networks 1 and 2 share the market, and all customers pay an average call price of $P = (p + q)/2$, since half of their calls are on-net and half of them are off-net. In this situation it does not pay for a customer to switch to network 3, since then all his calls would be off-net, with average price $q > (p + q)/2 = P$. Hence the state $(1/2, 1/2, 0)^t$ constitutes a Nash equilibrium. However, it turns out that all four mixed equilibria are unstable. The instability of a partially mixed equilibrium follows from the analysis of the two-networks case. By the positive tariff-mediated network externality, a small deviation from equilibrium creates a positive feedback loop leading one of the networks to cover the market. The dynamics in this case are illustrated in Figure 5.1. The same argument works for the completely mixed equilibrium, and hence we obtain the following result.

**Theorem 1** If the on-net price is below the off-net price, then in the long run a single network covers the market.

### 5.5.2 On-net Price above Off-net Price

Next we analyze the reverse case. Again we assume that $p_{ii} = p$ and $p_{ij} = q$ for all $i, j \in \{1, 2, 3\}$ and $i \neq j$. This time, however, suppose $p > q$, i.e. it
Figure 5.1: With positive tariff-mediated network externalities, any network can corner the market.

is cheaper to call off-net. With this price structure, it is clear that no single network can corner the market. If the market share of a network is close to one, almost all calls are on-net, and hence it pays to switch to another network, profiting from the low off-net price. Again we have the completely mixed symmetric equilibrium \( \bar{x} \) in the interior of the simplex. As opposed to the case above, this equilibrium is now unique. To see this, consider again the state \((1/2, 1/2, 0)^t\), where networks 1 and 2 share the market, and all customers pay an average call price of \( P = (p+q)/2 \). This time customers benefit from switching to network 3, since then all calls are off-net, with average price \( q < (p + q)/2 = P \). Hence the state \((1/2, 1/2, 0)^t\) is not a Nash equilibrium, and neither are the other two partially mixed symmetric states. The completely mixed equilibrium is globally asymptotically stable., see Figure 5.2. Analogous to the two-networks case, an on-net price above the off-net price creates negative tariff-mediated network externalities. The resulting negative feedback on market shares always makes the smallest network grow. This can be shown as follows. Consider any state \( x \). A customer of network \( i \) pays an average price of \( P_i = px_i + q(x_j + x_k) \), with \( \{i, j, k\} = \{1, 2, 3\} \), which can also be written as \( P_i = px_i + q(1 - x_i) = q + (p - q)x_i \). Since \( p > q \), \( P_i \) is increasing in \( x_i \). This means that the lowest average price is offered by the
smallest network. With customers beginning to switch to this network, its market share grows until another network becomes the smallest. The process comes to a halt if and only if all three networks are of the same size, i.e. if $x = \bar{x}$. This proves our next result.

**Theorem 2** If the on-net price is above the off-net price, then in the long run the three networks share the market equally.

### 5.5.3 Cyclic Symmetry

The third case we analyze here is more difficult. We keep the symmetry assumption, but this time we assume cyclic symmetry. Consider the following pricing structure: Network $i$ charges an on-net price of $p$, an off-net price of $q$ for calls to network $i + 1$ (where indices are counted modulo 3), and an off-net price of $r$ for calls to network $i + 2$. We assume the ordering $r < p < q$ of these prices. This means that for customers of network $i$ it is cheaper to call on-net than to call to network $i + 1$, but on-net calls are more expensive than calls to network $i - 1$. 

![Figure 5.2: With negative tariff-mediated network externalities, no network can corner the market.](image-url)
Figure 5.3: The vector of market shares cycles around the completely mixed equilibrium.

Note that this kind of pricing leads to a cyclic best response structure. The resulting matrix game is a variant of what is called the *Rock-Scissors-Paper game*, since its best response structure is the same as in the well-known children's game where one of the three symbols is shown by two players simultaneously, and rock beats scissors, which beats paper, which wins against rock.

It is straightforward in this case to show that the symmetric state $\bar{x}$ is the unique equilibrium. No single network $i$ can cover the market, since then all customers would strictly prefer to switch to network $i + 1$. Two networks cannot share the market. E.g. in state $(x_1, x_2, 0)^t$, customers of network 1 face an average price between $p$ and $q$, while customers of network 2 pay an average between $r$ and $p$. Since the latter is smaller, network 1 customers would always prefer network 2 over network 1. Note that this need not mean that they switch to network 2. If $x_2$ is small enough, customers prefer network 3 over both 1 and 2. The most interesting question here concerns the stability of the completely mixed equilibrium. As we have seen above, the cyclic symmetry of the call prices induces all best response paths to surround $\bar{x}$, following the best response cycle $1 \rightarrow 3 \rightarrow 2 \rightarrow 1$, as shown in Figure 5.3. Since the state space is a plane, and since best response paths...
do not intersect, a single path off the equilibrium can either spiral inwards or outwards, or form a closed cycle. If it forms a closed cycle, then this cycle consists of linear pieces, i.e. it is a polygon in the simplex, in our case a triangle. Such closed paths have first been found by Shapley (1964), and have therefore been termed *Shapley polygons* by Gaunersdorfer and Hofbauer (1995). In following their analysis, we can state the next result.

**Theorem 3** With cyclically symmetric prices, networks share the market equally in the long run if and only if $p - r \geq q - p$. If this is not the case, the market shares converge to a unique Shapley triangle.

Indeed, Gaunersdorfer and Hofbauer show that this theorem can be extended to more general price structures. Suppose network $i$ charges on-net price $p_i$ and off-net prices $q_i$ to network $i + 1$ and $r_i$ to network $i + 2$, where $r_{i+2} < p_{i+1} < q_i$ for all $i$. Then there exists a unique, completely mixed Nash equilibrium, and the following generalization of Theorem 3 holds.

**Theorem 4** The completely mixed Nash equilibrium is globally asymptotically stable if and only if

$$(p_1 - r_2)(p_2 - r_3)(p_3 - r_1) \geq (q_3 - p_1)(q_1 - p_2)(q_2 - p_3).$$

In case of instability, the market shares converge to a unique Shapley triangle.

The case of instability offers a new phenomenon, which is impossible to occur with only two networks. Irrespective of the initial state, in the long run the vector of market shares approaches a Shapley triangle in the interior of the state space, cycling along this triangle forever. Hence at any point in time all three networks are present in the market, and only their market shares change continually. The size of the Shapley triangle depends on the price differences. If $p - r$ is only a bit lower than $q - p$, the Shapley triangle is very small, and market shares are always close to equilibrium in the long run. If, however, $q - p$ is much larger than $p - r$, the Shapley triangle is close to the boundary triangle of the simplex, and each network is sometimes close to covering the market, before eventually the next network takes over. This phenomenon impressively demonstrates how careful one must be when interpreting equilibrium in a strategic context. In the static version of the game, traditional comparative static analysis would in any case treat the market as being in equilibrium.
5.5.4 The General Case

The last sections have been devoted to very special payoff structures. With more general payoff structures there are many different modes of market share behaviors. We can ask the question under what circumstances coexistence of all three networks is possible in the long run. One possibility is that an asymptotically stable shared market equilibrium exists. As we have seen, however, coexisting networks need not be in equilibrium, so we must take into account the possibility of an asymptotically stable Shapley polygon.

It can be shown that the existence of a Shapley polygon always implies the existence of a completely mixed equilibrium in the interior of the polygon. Indeed, let $L_1$ be an asymptotically stable Shapley polygon in $S_3$ and let $x$ be a point in the interior of $L_1$. Consider the $\alpha$-limit of the best response path $x(t)$ through $x(0) = x$, i.e. the set of limit points of $x(t)$ for $t \to -\infty$. This limit is well defined since the area enclosed by the Shapley polygon is backwards invariant. Since the backwards best response path cannot cross itself, it must converge to another limit cycle, i.e. a Shapley polygon $L_2$. Continuing in this fashion we can construct a sequence of Shapley polygons $L_1, L_2, L_3, \ldots$, where $L_{k+1}$ lies in the interior of $L_k$. If the sequence is infinite, then it must converge to a single point, which is a completely mixed equilibrium. If the sequence is finite, it must end in a degenerate Shapley polygon with empty interior, i.e. in a point. Again this point is necessarily a completely mixed equilibrium.

From the convexity of the different best response regions it follows that all Shapley polygons are Shapley triangles. However, actually the procedure suggested above always ends in the completely mixed equilibrium after one step, because Shapley triangles turn out to be unique here. An elegant argument from projective geometry establishes this as follows (see Gaunersdorfer and Hofbauer, 1995). Suppose there are two Shapley polygons. Their vertices lie pairwise on the lines where $(Ax)_i = (Ax)_j$, and these three lines intersect in the completely mixed equilibrium $\bar{x}$. Hence the vertices of the triangles are perspective from $\bar{x}$. By Desargues Theorem, the extensions of their edges must pairwise intersect in three collinear points. However, this yields a contradiction, since the intersection points are the vertices of the simplex $S_3$, which are obviously not collinear.
The arguments from the last two paragraphs establish that coexistence of all three networks is possible if and only if there exists an asymptotically stable completely mixed Nash equilibrium or an asymptotically stable Shapley triangle. In the latter case there exists an unstable completely mixed equilibrium in the interior of the triangle. Hence existence of a completely mixed equilibrium \( \bar{x} \) is a necessary (but not sufficient) condition for coexistence. In this equilibrium, the payoffs are the same for all three pure strategies, i.e. \((A\bar{x})_1 = (A\bar{x})_2 = (A\bar{x})_3\), and the equilibrium is determined by the intersection of the three indifference halflines \( l_{12}, l_{23}, \) and \( l_{31} \). The best response sets are convex polytopes bounded by these lines. Consider now a small deviation from \( \bar{x} \) in the direction of network 1, i.e. \( e_1 \). The new state can be written as a convex combination

\[
x^1_\varepsilon = \varepsilon e_1 + (1 - \varepsilon)\bar{x}
\]  

for some small \( \varepsilon > 0 \). If \( e_1 \) is a Nash equilibrium, i.e. if \( e_1 \in BR(e_1) \), then by convexity of the best response sets and linearity of payoffs, \( e_1 \in BR(x^1_\varepsilon) \) for all \( 1 \geq \varepsilon \geq 0 \). As a consequence, there is a best response path starting from \( \bar{x} \) and converging to \( e_1 \). In this case the completely mixed equilibrium is unstable and a Shapley triangle does not exist. This contradicts our assumption, hence we must have \( e_1 \notin BR(e_1) \). Then either \( e_2 \) or \( e_3 \) are best responses to \( e_1 \).

Next consider a small deviation from \( \bar{x} \) in the direction of \( e_2 \). As before, the new state can be written as a convex combination \( x^2_\varepsilon = \varepsilon e_2 + (1 - \varepsilon)\bar{x} \) for some small \( \varepsilon > 0 \). Again we conclude that \( e_2 \notin BR(e_2) \), implying that either \( e_1 \) or \( e_3 \) are best responses to \( e_2 \). A completely analogous argument establishes that either \( e_1 \) or \( e_2 \) are best responses to \( e_3 \).

Assume \( x(s) \) for \( s \in [0, 1] \) is a path (not a best response path) on the boundary of \( S_3 \), starting at \( e_1 \) and moving clockwise to \( e_2 \), to \( e_3 \), and back to \( e_1 \). Let \( b = (i, j, k) \) denote the sequence of networks which are best responses to \( x(s) \) along this path. Since all three networks must be present in the sequence (otherwise there would not be a completely mixed equilibrium), and since \( e_1 \) cannot be a best response to \( x(0) = e_1 \), we have the following four possible configurations:

\[
b \in \{(2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\}
\]  

(5.6)
We can w.l.o.g. renumber the networks in such a way that \( e_2 \in BR(e_1) \). This leaves us with the two possibilities

\[
b = (2, 1, 3) \quad \text{or} \quad b = (2, 3, 1)
\]

(5.7)

Consider the first case, \( b = (2, 1, 3) \). Here the first switch of the best response to \( x(s) \) is from \( e_2 \) to \( e_1 \), and it occurs at the intersection of the boundary of the simplex with the indifference halfline \( l_{12} \). Since \( e_2 \) is not a Nash equilibrium, the indifference halfline \( l_{12} \) must intersect the edge connecting \( e_1 \) and \( e_2 \). Call the intersection point \( x^{12} \). In state \( x^{12} \) agents are subscribed only to networks 1 and 2, they are indifferent between these two networks and strictly prefer them to network 3. Hence \( x^{12} \) is a Nash equilibrium. Again it follows that there is a best response path leaving \( \bar{x} \) and converging to \( x^{12} \), rendering \( \bar{x} \) unstable and implying the nonexistence of a Shapley triangle. This is a contradiction, so \( b \neq (2, 1, 3) \). It follows that \( b = (2, 3, 1) \).

The last paragraph proved that \( b = (2, 3, 1) \) is a necessary condition for coexistence. We show now that together with the nonexistence of pure strategy equilibria it is also sufficient.

Note that \( b = (2, 3, 1) \) together with the nonexistence of a pure strategy equilibrium effectively means that for any two networks \( i \) and \( j \), the indifference halfline \( l_{ij} \) does not intersect the edge connecting \( e_i \) and \( e_j \). This is equivalent to saying that there are no Nash equilibria on the edges of the simplex. In this case the boundary of the simplex is repelling for best response path, and every such path eventually stays in the interior of the simplex. Hence, nonexistence of a boundary Nash equilibrium is a sufficient condition for coexistence. The same is of course true in the two networks case. We subsume this analysis in the following theorem.

**Theorem 5** In the case of at most three networks there is coexistence in the long run if and only if there are no Nash equilibria involving an unused network.

Note that the analysis relies on special geometric properties of the plane which do not extend to higher dimensions. For this reason we cannot derive the same result for the case of four or more networks.
5.6 Discussion

The emphasis in this chapter lies on the point that the usual static interpretation of consumer equilibrium has a serious weakness in models of network competition. We have seen that as soon as three or more networks are present, existence of a stable consumer equilibrium is no longer guaranteed. Moreover, if such an equilibrium happens to exist, then there are typically multiple ones. An exception is the case we called on-net prices above off-net prices, where a unique and globally asymptotically stable shared market equilibrium exists. Unfortunately the pricing patterns observed in real-world telecommunications networks are just the opposite of this case. This leads to an interesting puzzle, to which the next chapter is devoted.