3 Tableau system for Term Logic

3.1 Introductory remarks

Now, we will describe the tableau system for Term Logic[1] (for short: **TL**).[2] By Term Logic we mean a logic in which both premises and conclusions have the form of classical categorical propositions:[3]

- Each *P* is *Q*.
- Some *P* is *Q*.
- No *P* is *Q*.
- Some *P* is not *Q*.

Moreover, we do not assume that in Term Logic the names appearing in categorical propositions are not empty. So we consider the most general approach — the simplest language and semantics.

The tableau system we will describe can be treated as a stand-alone system. But, this is not the purpose of its construction. We intend to indicate an example of the use of analogous tableau concepts, such as those we defined for the tableau system for **CPL**. Although the defined system will also feature the property of a finite branch, there will be new features that will be mentioned soon.

We will redefine the concepts of rule, various types of branches and tableaux in an almost identical way to the tableau concepts defined in the previous chapter. We mean *almost identical* because, after all, we will face a different language of tableau proof and a different set of tableau rules than in the case of the tableau

---

1 Because we are going to name systems of various types of reasoning about the relationships between names, we will use rather term 'Term Logic' than 'Syllogistic logic'.

2 The considerations contained in this chapter are based on the English language study [9]. In that paper, we presented an outline of the tableau system for **TL** without a thorough analysis of details. Since the production of this article, we have also generalized the concept of tableau rules and modified other tableau concepts which has affected other concepts and the very nature of the tableau system. Some other variants of tableaux for syllogistic are presented in [15], [22].

3 We write about classic categorical propositions because categorical propositions can also take non-classical forms. They can be e.g. any numerical propositions: *At least five P are Q*, or e.g. modal propositions *de re*: *Each P is by necessity Q*. Besides, in the last chapter we will construct a tableau system for the logic of categorical modal propositions *de re*. In addition, there are many other possibilities to enrich classical categorical propositions.

system for **CPL**. However, these differences will not affect the formal nature of concepts themselves, so although, for example, a branch for **TL** will be built from different sets of expressions than a branch for **CPL**, the structure of presentation itself will be identical — because we aim at presenting a general scheme of the tableau system construction, a synthesis consisting in abstracting from tableau concepts those properties that are not specific, and therefore do not depend on the characteristics of exemplary systems which we define in detail in the initial chapters of the book.

The importance of the presented tableau system **TL** for our considerations consists in the fact that the set of formulas for Term Logic is a proper subset of the set of proof expressions, and moreover, there are no branchings in the tableaux. This case is located between borderline cases because of the relation of the set of logic formulas to the set of tableau expressions. However, in some respects it has a borderline character itself, because the tableaux do not host any branchings, so the tableau proofs boil down to the construction of a single maximal branch. General tableau concepts could be therefore simplified in the tableau system for **TL** although they still provide more detail on the general concepts that we describe in the book.

3.2 Language and semantics

The construction of tableau system for Term Logic requires, as usual, the presentation of basic concepts. Let us start with the alphabet of language of **TL**.

Definition 3.1 (Alphabet of **TL**). The alphabet of Term Logic is the sum of the following sets:

- set of logical constants: $\mathsf{Lc} = \{\mathbf{a}, \mathbf{i}, \mathbf{e}, \mathbf{o}\}$
- set of name letters: $\mathsf{Ln} = \{P^1, Q^1, R^1, P^2, Q^2, R^2, \dots\}$.

Although the set of name letters is infinite and includes indexed letters, in practice we will use a finite number of the following letters: P, Q, R, S, T, U, treating them as metavariables ranging over set Ln.

Let us now proceed to the definition of formula of **TL**.

Definition 3.2 (Formula of **TL**). Set of formulas **TL** is the smallest set containing the following expressions:

- $P\mathbf{a}Q$
- $P\mathbf{i}Q$
- $P\mathbf{e}Q$
- $P\mathbf{o}Q$

where $P, Q \in \mathsf{Ln}$.
We specify this set as $\mathsf{For}_{\mathbf{TL}}$, and its elements will be called *formulas*.

Another basic concept is the concept of model for **TL**, and then the concept of truth in the model.

Definition 3.3 (Model for language of **TL**). *Model* $\mathfrak{M}_{\mathbf{TL}}$ *for language of* **TL** will be called such ordered pair $\langle D, d \rangle$ that:

- D is any set
- d is a function from set Ln in set $P(D)$ of all subsets of set D, i.e. $d : \mathsf{Ln} \longrightarrow P(D)$.

Definition 3.4 (Truth in model). Let $\mathfrak{M}_{\mathbf{TL}} = \langle D, d \rangle$ be a model and $A \in \mathsf{For}_{\mathbf{TL}}$. We shall state that formula A is *true in model* $\mathfrak{M}_{\mathbf{TL}}$ (for short $\mathfrak{M}_{\mathbf{TL}} \vDash A$) iff for some name letters P, $Q \in \mathsf{Ln}$, one of the below conditions is met:

1. $A := P\mathbf{a}Q$ and $d(P) \subseteq d(Q)$
2. $A := P\mathbf{i}Q$ and $d(P) \cap d(Q) \neq \varnothing$
3. $A := P\mathbf{e}Q$ and $d(P) \cap d(Q) = \varnothing$
4. $A := P\mathbf{o}Q$ and $d(P) \nsubseteq d(Q)$.

Formula A is *false in model* $\mathfrak{M}_{\mathbf{TL}}$ (for short $\mathfrak{M}_{\mathbf{TL}} \nvDash A$) if for any name letters P, $Q \in \mathsf{Ln}$ none of the conditions is met.

Let $X \subseteq \mathsf{For}_{\mathbf{TL}}$. Set X is *true in model* $\mathfrak{M}_{\mathbf{TL}}$ (for short: $\mathfrak{M}_{\mathbf{TL}} \vDash X$) iff for any formula $A \in X$, $\mathfrak{M}_{\mathbf{TL}} \vDash A$. Set of formulas X is *false in model* $\mathfrak{M}_{\mathbf{TL}}$ (for short: $\mathfrak{M}_{\mathbf{TL}} \nvDash X$) iff it is not the case that for any formula $A \in X$, $\mathfrak{M}_{\mathbf{TL}} \vDash A$.

Making use of the concept of model, we can now define the concept of entailment or otherwise semantic consequence relation in **TL**.

Definition 3.5 (Semantic consequence of **TL**). Let $X \subseteq \mathsf{For}_{\mathbf{TL}}$ and $A \in \mathsf{For}_{\mathbf{TL}}$. From set X *follows* formula A (for short: $X \vDash A$) iff for any model $\mathfrak{M}_{\mathbf{TL}}$, if $\mathfrak{M}_{\mathbf{TL}} \vDash X$, then $\mathfrak{M}_{\mathbf{TL}} \vDash A$. Relation $\vDash$ will be also called *semantic consequence relation* of Term Logic, or shortly *semantic consequence*.

Denotation 3.6. For any set of formulas X and any formula A, $X \nvDash A$ will mean that it is not the case that $X \vDash A$.

We will now take up the issue of the compactness of semantic consequence relation. According to definition 2.48 compactness of relation $\vDash$ is expressed by the following definition.

Definition 3.7 (Compactness of semantic consequence of **TL**). Relation of semantic consequence $\vDash$ is *compact* iff for any set of formulas X and any formula A it is the case that: $X \vDash A \Leftrightarrow$ there exists such finite set $Y \subseteq X$ that $Y \vDash A$.

Compactness of relation $\models$ seems pretty obvious, so we are just going to outline the proof consisting in embedding **TL** into Monadic Logic of Predicates of which the relation of consequence is compact.

Before we proceed to the verbalization and proof of the relevant theorem, let us recall a few concepts concerning the Monadic Logic of Predicates (for short **MLP**):

- the alphabet of **MLP** contains:
  - classical, Boolean constants: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, and quantifiers $\exists$, $\forall$
  - unary predicate letters $\mathsf{Lp} = \{p_1, q_1, r_1, p_2, q_2, r_2, \ldots\}$ (in practice, we will use a finite number of the following letters: *p*, *q*, *r*, *s*, treating them as metavariables ranging over set Lp)
  - set of individual constants Ci and individual variables Vi as well as auxiliary symbols:), (.
- **MLP** formulas will be constructed in a standard way and these are atomic expressions of type $p(x)$, where $p \in \mathsf{Lp}$, whereas $x \in \mathsf{Ci} \cup \mathsf{Vi}$, and more complex expressions using quantifiers, Boolean constants and brackets; set of formulas **MLP** we shall denote as $\mathsf{For}_{\mathbf{MLP}}$
- models for formulas from set $\mathsf{For}_{\mathbf{MLP}}$ are ordered triples $\mathfrak{M}_{\mathbf{MLP}} = \langle D, d_{\mathsf{Lp}}, d_{\mathsf{Ci}} \rangle$, where:
  - D is a non-empty set of any objects which is called a domain
  - d_{Lp} is a function from the set of predicate letters in the set of all subsets of D, i.e. in $P(D)$
  - d_{Ci} is a function from the set of individual constants in set D
- both truth conditions for **MLP** formulas and relation of semantic consequence $\models_{\mathbf{MLP}}$ are defined in a standard way
- relation $\models_{\mathbf{MLP}}$ is compact.

Since set of name letters Ln set of predicate letters Lp are countable, then there exists bijection: $\pi : \mathsf{Ln} \longrightarrow \mathsf{Lp}$, where $\pi(X) = x$, for all letters. Obviously, for π there exists the inverse function π^{-1}.

Next, we define function g from set of formulas $\mathsf{For}_{\mathbf{TL}}$ in set of formulas $\mathsf{For}_{\mathbf{MLP}}$ with the following conditions, for any name letters $P, Q \in \mathsf{Ln}$:

1. $g(P\mathbf{a}Q) = \forall_x(\pi(P)(x) \rightarrow \pi(Q)(x))$
2. $g(P\mathbf{i}Q) = \exists_x(\pi(P)(x) \wedge \pi(Q)(x))$
3. $g(P\mathbf{e}Q) = \forall_x(\pi(P)(x) \rightarrow \neg\pi(Q)(x))$
4. $g(P\mathbf{o}Q) = \exists_x(\pi(P)(x) \wedge \neg\pi(Q)(x))$,

where x is any, but fixed variable from set Vi.

In turn, having function g, we define transformation $\textsc{Tr}\colon \mathsf{For}_{\mathbf{TL}} \longrightarrow g(\mathsf{For}_{\mathbf{TL}})$ in such way that $\textsc{Tr}(y) = g(y)$, for any $y \in \mathsf{For}_{\mathbf{TL}}$. Note that function $\textsc{Tr}$ is a bijection because:

(a) for each $y \in g(\mathsf{For}_{\mathbf{TL}})$ there exists such $z \in \mathsf{For}_{\mathbf{TL}}$ that $\textsc{Tr}(z) = y$

(b) for any $y, z \in \mathsf{For}_{\mathbf{TL}}$, if y and z are various formulas, then $\textsc{Tr}(y) \neq \textsc{Tr}(z)$, by definition of function g.

Hence, there exists the inverse function to $\textsc{Tr}$, function $\textsc{Tr}^{-1}\colon g(\mathsf{For}_{\mathbf{TL}}) \longrightarrow \mathsf{For}_{\mathbf{TL}}$, so such function that for any $y \in \mathsf{For}_{\mathbf{TL}}$, $\textsc{Tr}^{-1}(\textsc{Tr}(y)) = y$.

Now, we will show that the following fact holds.

Proposition 3.8. *For any set of formulas $X \subseteq \mathsf{For}_{\mathbf{TL}}$ and any formula A: $X \vDash A \Leftrightarrow \textsc{Tr}(X) \vDash_{\mathbf{MLP}} \textsc{Tr}(A)$.*

Proof. Take any set of formulas $X \subseteq \mathsf{For}_{\mathbf{TL}}$ and formula A.

First, let us consider implication '$\Rightarrow$', assuming that $X \vDash A$. Take any model $\mathfrak{M}_{\mathbf{MLP}} = \langle D, d_{\mathsf{Lp}}, d_{\mathsf{Ci}} \rangle$ such that $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \textsc{Tr}(X)$. Based on model $\mathfrak{M}_{\mathbf{MLP}}$ we will define model $\mathfrak{M}_{\mathbf{TL}} = \langle D', d \rangle$ as follows:

- $D' = D$
- for any $P \in \mathsf{Ln}$, $d(P) = d_{\mathsf{Lp}}(\pi(P))$.

We will show that $\mathfrak{M}_{\mathbf{TL}} \vDash X$. We will now consider cases of formulas that can belong to set X. Take any name letters $P, Q \in \mathsf{Ln}$ and assume that some of the cases occurs. We know that for some $p, q \in \mathsf{Lp}$, $\pi(P) = p$ and $\pi(Q) = q$.

1. $P\mathbf{a}Q \in X$, then $\textsc{Tr}(P\mathbf{a}Q) = \forall_x(p(x) \rightarrow q(x))$ and by assumption $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \forall_x(p(x) \rightarrow q(x))$, consequently, for each denotation of variable x, if denotation x belongs to set $d_{\mathsf{Lp}}(p)$, then it belongs to set $d_{\mathsf{Lp}}(q)$, hence by definition of model $\mathfrak{M}_{\mathbf{TL}}$, $d(P) \subseteq d(Q)$, so by definition of truth in model 3.4, $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{a}Q$
2. $P\mathbf{i}Q \in X$, then $\textsc{Tr}(P\mathbf{i}Q) = \exists_x(p(x) \wedge q(x))$ and by assumption $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \exists_x(p(x) \wedge q(x))$, hence there exists such denotation of variable x that this denotation belongs to set $d_{\mathsf{Lp}}(p)$ and set $d_{\mathsf{Lp}}(q)$, thus by definition of model $\mathfrak{M}_{\mathbf{TL}}$, $d(P) \cap d(Q) \neq \varnothing$, so by definition of truth in model 3.4, $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{i}Q$
3. $P\mathbf{e}Q \in X$, then $\textsc{Tr}(P\mathbf{e}Q) = \forall_x(p(x) \rightarrow \neg q(x))$ and by assumption $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \forall_x(p(x) \rightarrow \neg q(x))$, hence for each denotation of variable x, if denotation x belongs to set $d_{\mathsf{Lp}}(p)$, then it does not belong to set $d_{\mathsf{Lp}}(q)$, thus by definition of model $\mathfrak{M}_{\mathbf{TL}}$, $d(P) \cap d(Q) = \varnothing$, so by definition of truth in model 3.4, $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{e}Q$

4. $P\mathbf{o}Q \in X$, then $\textsc{Tr}(P\mathbf{o}Q) = \exists_x(p(x) \wedge \neg q(x))$ and by assumption $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \exists_x(p(x) \wedge \neg q(x))$, hence there exists such denotation of variable x that this denotation belongs to set $d_{\mathsf{Lp}}(p)$ and does not belong to set $d_{\mathsf{Lp}}(q)$, thus by definition of model $\mathfrak{M}_{\mathbf{TL}}$, $d(P) \nsubseteq d(Q)$, so by definition of truth in model 3.4, $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{o}Q$.

Hence, $\mathfrak{M}_{\mathbf{TL}} \vDash X$. In turn, from definition of relation $\vDash$ it follows that $\mathfrak{M}_{\mathbf{TL}} \vDash A$.

Now, we will show that $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \textsc{Tr}(A)$. We will consider cases of formulas that can be identical to formula A. Take any name letters $P, Q \in \mathsf{Ln}$ and assume that some of the cases occurs. We know that for some $p, q \in \mathsf{Lp}$, $\pi(P) = p$ and $\pi(Q) = q$.

1. $A = P\mathbf{a}Q$, then $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{a}Q$, so by definition of truth in model 3.4, $d(P) \subseteq d(Q)$, thus by definition of model $\mathfrak{M}_{\mathbf{TL}}$, for each denotation of variable x, if denotation x belongs to set $d_{\mathsf{Lp}}(p)$, then it belongs to set $d_{\mathsf{Lp}}(q)$, hence $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \forall_x(p(x) \rightarrow q(x))$, thus $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \textsc{Tr}(P\mathbf{a}Q)$
2. $A = P\mathbf{i}Q$, then $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{i}Q$, so by definition of truth in model 3.4, $d(P) \cap d(Q) \neq \varnothing$, thus by definition of model $\mathfrak{M}_{\mathbf{TL}}$, there exists such denotation of variable x that this denotation belongs to set $d_{\mathsf{Lp}}(p)$ and to set $d_{\mathsf{Lp}}(q)$, hence $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \exists_x(p(x) \wedge q(x))$, thus $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \textsc{Tr}(P\mathbf{i}Q)$
3. $A = P\mathbf{e}Q$, then $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{e}Q$, so by definition of truth in model 3.4, $d(P) \cap d(Q) = \varnothing$, thus by definition of model $\mathfrak{M}_{\mathbf{TL}}$, for each denotation of variable x, if denotation x belongs to set $d_{\mathsf{Lp}}(p)$, then it does not belong to set $d_{\mathsf{Lp}}(q)$, so $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \forall_x(p(x) \rightarrow \neg q(x))$, thus $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \textsc{Tr}(P\mathbf{e}Q)$
4. $A = P\mathbf{o}Q$, then $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{o}Q$, so by definition of truth in model 3.4, $d(P) \nsubseteq d(Q)$, thus by definition of model $\mathfrak{M}_{\mathbf{TL}}$, there exists such denotation of variable x, that this denotation belongs to set $d_{\mathsf{Lp}}(p)$, but it does not belong to set $d_{\mathsf{Lp}}(q)$, so $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \exists_x(p(x) \wedge \neg q(x))$, thus $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \textsc{Tr}(P\mathbf{o}Q)$.

Hence, $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \textsc{Tr}(A)$. Whereas from the arbitrariness of model $\mathfrak{M}_{\mathbf{MLP}}$ we have $\textsc{Tr}(X) \vDash_{\mathbf{MLP}} \textsc{Tr}(A)$.

Next, let us consider implication '$\Leftarrow$', assuming that $\textsc{Tr}(X) \vDash_{\mathbf{MLP}} \textsc{Tr}(A)$. Take any model $\mathfrak{M}_{\mathbf{TL}} = \langle D, d \rangle$ such that $\mathfrak{M}_{\mathbf{TL}} \vDash X$. Based on model $\mathfrak{M}_{\mathbf{TL}}$, we will define model $\mathfrak{M}_{\mathbf{MLP}} = \langle D', d_{\mathsf{Lp}}, d_{\mathsf{Ci}} \rangle$ as follows:

- $D' = D$, if $D \neq \varnothing$; otherwise $D' = D''$, for some whichever, fixed $D'' \neq \varnothing$
- for any $p \in \mathsf{Lp}$, $d_{\mathsf{Lp}}(p) = d(\pi^{-1}(p))$
- for any $x \in \mathsf{Ci}$, $d_{\mathsf{Ci}}(x) \in D'$.

We will show that $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \textsc{Tr}(X)$. We will now consider cases of formulas that can belong to set X, and consequently, their images under function $\textsc{Tr}$ belong to $\text{Tr}(X)$. Take any name letters $P, Q \in \mathsf{Ln}$ and assume that some of the below cases occurs. We know that for some $p, q \in \mathsf{Lp}$, $\pi(P) = p$ and $\pi(Q) = q$.

1. $P\mathbf{a}Q \in X$, then $\mathfrak{M}_{\mathbf{TL}} \models P\mathbf{a}Q$, so by definition of truth in model 3.4, $d(P) \subseteq d(Q)$, thus by definition of model $\mathfrak{M}_{\mathbf{MLP}}$, for each denotation of variable x, if denotation x belongs to set $d_{\mathsf{Lp}}(p)$, then it belongs to set $d_{\mathsf{Lp}}(q)$, hence $\mathfrak{M}_{\mathbf{MLP}} \models_{\mathbf{MLP}} \forall_x(p(x) \to q(x))$, thus $\mathfrak{M}_{\mathbf{MLP}} \models_{\mathbf{MLP}} \text{Tr}(P\mathbf{a}Q)$
2. $P\mathbf{i}Q \in X$, then $\mathfrak{M}_{\mathbf{TL}} \models P\mathbf{i}Q$, so by definition of truth in model 3.4, $d(P) \cap d(Q) \neq \varnothing$, thus by definition of model $\mathfrak{M}_{\mathbf{MLP}}$, there exists such denotation of variable x, that this denotation belongs to set $d_{\mathsf{Lp}}(p)$ and to set $d_{\mathsf{Lp}}(q)$, hence $\mathfrak{M}_{\mathbf{MLP}} \models_{\mathbf{MLP}} \exists_x(p(x) \wedge q(x))$, thus $\mathfrak{M}_{\mathbf{MLP}} \models_{\mathbf{MLP}} \text{Tr}(P\mathbf{i}Q)$
3. $P\mathbf{e}Q \in X$, then $\mathfrak{M}_{\mathbf{TL}} \models P\mathbf{e}Q$, so by definition of truth in model 3.4, $d(P) \cap d(Q) = \varnothing$, thus by definition of model $\mathfrak{M}_{\mathbf{MLP}}$, for each denotation of variable x, if denotation x belongs to set $d_{\mathsf{Lp}}(p)$, then it does not belong to set $d_{\mathsf{Lp}}(q)$, hence $\mathfrak{M}_{\mathbf{MLP}} \models_{\mathbf{MLP}} \forall_x(p(x) \to \neg q(x))$, thus $\mathfrak{M}_{\mathbf{MLP}} \models_{\mathbf{MLP}} \text{Tr}(P\mathbf{e}Q)$
4. $P\mathbf{o}Q \in X$, then $\mathfrak{M}_{\mathbf{TL}} \models P\mathbf{o}Q$, so by definition of truth in model 3.4, $d(P) \nsubseteq d(Q)$, thus by definition of model $\mathfrak{M}_{\mathbf{MLP}}$, there exists such denotation of variable x that this denotation belongs to set $d_{\mathsf{Lp}}(p)$, but it does not belong to set $d_{\mathsf{Lp}}(q)$, hence $\mathfrak{M}_{\mathbf{MLP}} \models_{\mathbf{MLP}} \exists_x(p(x) \wedge \neg q(x))$, thus $\mathfrak{M}_{\mathbf{MLP}} \models_{\mathbf{MLP}} \text{Tr}(P\mathbf{o}Q)$.

Hence, $\mathfrak{M}_{\mathbf{MLP}} \models_{\mathbf{MLP}} \text{Tr}(X)$. From definition of relation $\models_{\mathbf{MLP}}$ it follows that $\mathfrak{M}_{\mathbf{MLP}} \models_{\mathbf{MLP}} \text{Tr}(A)$.

We will now show that $\mathfrak{M}_{\mathbf{TL}} \models A$. We will consider cases of formulas that can be identical to formula A. Take any name letters $P, Q \in \mathsf{Ln}$ and assume that some of the below cases occurs. We know that for some $p, q \in \mathsf{Lp}$, $\pi(P) = p$ and $\pi(Q) = q$.

1. $A := P\mathbf{a}Q$, then $\text{Tr}(P\mathbf{a}Q) = \forall_x(p(x) \to q(x))$, and since $\mathfrak{M}_{\mathbf{MLP}} \models_{\mathbf{MLP}} \forall_x(p(x) \to q(x))$, consequently, for each denotation of variable x, if denotation x belongs to set $d_{\mathsf{Lp}}(p)$, then it belongs to set $d_{\mathsf{Lp}}(q)$, hence by definition of model $\mathfrak{M}_{\mathbf{MLP}}$, $d(P) \subseteq d(Q)$, so by definition of truth in model 3.4, $\mathfrak{M}_{\mathbf{TL}} \models P\mathbf{a}Q$
2. $A := P\mathbf{i}Q$, then $\text{Tr}(P\mathbf{i}Q) = \exists_x(p(x) \wedge q(x))$, and because $\mathfrak{M}_{\mathbf{MLP}} \models_{\mathbf{MLP}} \exists_x(p(x) \wedge q(x))$, hence there exists such denotation of variable x that this denotation belongs to set $d_{\mathsf{Lp}}(p)$ and set $d_{\mathsf{Lp}}(q)$, thus by definition of model $\mathfrak{M}_{\mathbf{MLP}}$, $d(P) \cap d(Q) \neq \varnothing$, so by definition of truth in model 3.4, $\mathfrak{M}_{\mathbf{TL}} \models P\mathbf{i}Q$
3. $A := P\mathbf{e}Q$, then $\text{Tr}(P\mathbf{e}Q) = \forall_x(p(x) \to \neg q(x))$, and because $\mathfrak{M}_{\mathbf{MLP}} \models_{\mathbf{MLP}} \forall_x(p(x) \to \neg q(x))$, hence for each denotation of variable x, if denotation x belongs to set $d_{\mathsf{Lp}}(p)$, then it does not belong to set $d_{\mathsf{Lp}}(q)$, thus by definition of model $\mathfrak{M}_{\mathbf{MLP}}$, $d(P) \cap d(Q) = \varnothing$, so by definition of truth in model 3.4, $\mathfrak{M}_{\mathbf{TL}} \models P\mathbf{e}Q$

4. $A := P\mathbf{o}Q$, then $\text{Tr}(P\mathbf{o}Q) = \exists_x(p(x) \wedge \neg q(x))$, and because $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \exists_x(p(x) \wedge \neg q(x))$, hence there exists such denotation of variable x, that this denotation belongs to set $d_{\mathsf{Lp}}(p)$ and does not belong to set $d_{\mathsf{Lp}}(q)$, thus by definition of model $\mathfrak{M}_{\mathbf{MLP}}$, $d(P) \nsubseteq d(Q)$, so by definition of truth in model 3.4, $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{o}Q$.

Hence, $\mathfrak{M}_{\mathbf{TL}} \vDash A$. Whereas from the arbitrariness of model $\mathfrak{M}_{\mathbf{TL}}$ we have $X \vDash A$. □

Let us now proceed to the very fact concerning the compactness of relation $\vDash$.

Proposition 3.9. *Relation of semantic consequence $\vDash$ is compact.*

Proof. Now, take any set $X \subseteq \mathsf{For}_{\mathbf{TL}}$ and any formula $A \in \mathsf{For}_{\mathbf{TL}}$ and assume that $X \vDash A$. By virtue of fact 3.8 we know that $\text{Tr}(X) \vDash_{\mathbf{MLP}} \text{Tr}(A)$. And since relation $\vDash_{\mathbf{MLP}}$ is compact, then there exists such finite subset $Y' \subseteq \text{Tr}(X)$ that $Y' \vDash_{\mathbf{MLP}} \text{Tr}(A)$.

Due to definition of function Tr and fact 3.8, there exists such finite subset $Y \subseteq X$ that $\text{Tr}^{-1}(Y') = Y$ and $Y \vDash A$.

On the other hand, let us assume there exists such finite subset $Y \subseteq X$ that $Y \vDash A$. Then, however, due to definition of relation of semantic consequence of **TL** 3.5, $X \vDash A$.

Thus, relation of semantic consequence $\vDash$ of **TL** is compact. □

3.3 Basic concepts of the tableau system for TL

Unlike the tableau system for **CPL** in the case of tableau system for **TL** the tableau proofs will be carried out in a more rich language than the set of formulas. The elements of expressions of this language will be simply called *tableau expressions*.

Definition 3.10 (Tableau expressions of **TL**). *Set of tableau expressions* is the union of the three following sets:

- $\{P_{+i} : P \in \mathsf{Ln}, i \in \mathbb{N}\}$
- $\{P_{-i} : P \in \mathsf{Ln}, i \in \mathbb{N}\}$
- $\mathsf{For}_{\mathbf{TL}}$.

We specify this set as $\mathsf{Te}_{\mathbf{TL}}$, and its elements will be called *expressions* or *tableau expressions*. Numbers that exist in expressions with + or − sign will be called *indices*.

Remark 3.11. In case of **TL** set $\mathsf{Te}_{\mathbf{TL}}$, i.e. set of proof expressions of which subsets will be used for construction of tableaux, is composed of formulas of **TL** and additional expressions which play a role worth explaining. We mean expressions P_{+i}, P_{-i}, where $P \in \mathsf{Ln}$ and $i \in \mathbb{N}$. Although our approach to the tableau system is syntactic — we treat tableau proof as a transformation of sets of symbols without reference to their meaning — we can point to the semantic intuitions behind such kind of expressions. The natural numbers appearing in the expressions, in the construction of the model will denote the objects in the universe under consideration, while symbols +/− will mean being or not being the designator of a given name denoted by letter P. According to known literature, it seems that this sort of use of an additional language to describe whether or not given objects belong to the ranges of names in the context of tableau proofs has not yet been fully developed.

We will now define an auxiliary function that is to attribute formulas to formulas that contradict them. This function, among other things, will be used to begin tableau proofs, so it will play a similar role as negation in the tableau system from the previous chapter.

Definition 3.12. Function $\circ : \mathsf{For}_{\mathbf{TL}} \longrightarrow \mathsf{For}_{\mathbf{TL}}$, for any $P, Q \in \mathsf{Ln}$, is defined with the following conditions:

1. $\circ(P\mathbf{a}Q) = P\mathbf{o}Q$
2. $\circ(P\mathbf{i}Q) = P\mathbf{e}Q$
3. $\circ(P\mathbf{e}Q) = P\mathbf{i}Q$
4. $\circ(P\mathbf{o}Q) = P\mathbf{a}Q$.

Notice that by virtue of definition 3.12 and definition of truth in model 3.4, the following fact occurs.

Proposition 3.13. *For any formula A and any model $\mathfrak{M}_{TL}$: $\mathfrak{M}_{TL} \models A$ iff $\mathfrak{M}_{TL} \not\models \circ(A)$.*

As we wrote, one of the basic concepts used to describe a tableau system, due to the nature of tableau proofs, is the concept of a tableau inconsistent set of proof expressions. Let it be reminded that in case of the defined system for **TL**, the proof expressions are the proper superset of the set of formulas, so the concept of t-incosistent set of formulas will also cover additional expressions.

Definition 3.14 (Tableau inconsistent set of expressions)**.** Set $X \subseteq \mathsf{Te}_{\mathbf{TL}}$ will be calles *tableau inconsistent* (for short: t-inconsistent) iff one of the below conditions is met:

1. there exists such formula $A \in \mathsf{For}_{\mathbf{TL}}$ that $A \in X$ and $\circ(A) \in X$
2. there exists such name letter $P \in \mathsf{Ln}$ and such number $i \in \mathbb{N}$ that $P_{+i} \in X$ and $P_{-i} \in X$.

Set X will be called *t-consistent* iff it is not t-inconsistent.

Remark 3.15. From the definition of tableau inconsistent set of expressions 3.14 we might remove the first condition and require the t-inconsistency to emerge in the set of “pure” tableau expressions. However, we will leave this condition to find t-inconsistency faster wherever possible, without the need for further application of tableau rules.

Let us now introduce the concept of model appropriate for the set of expressions. It is a generalisation of the concept of truth in model in the entire set $\mathsf{Te}_{\mathbf{TL}}$.

Definition 3.16 (Model appropriate for the set of expressions)**.** Let X be a set of tableau expressions, while $\mathfrak{M}_{\mathbf{TL}} = \langle D, d \rangle$ be a model. Model $\mathfrak{M}_{\mathbf{TL}}$ is *appropriate* for set X iff the below conditions are met:

1. $\mathfrak{M}_{\mathbf{TL}} \vDash X \cap \mathsf{For}_{\mathbf{TL}}$
2. there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that for each name letter $P \in \mathsf{Ln}$ and each $i \in \mathbb{N}$:
   a. if $P_{+i} \in X$, then $\gamma(i) \in d(P)$
   b. if $P_{-i} \in X$, then $\gamma(i) \notin d(P)$.

From the two above definitions, an important conclusion for metatheory follows, namely the relationship between the inconsistent sets of expressions and the appropriateness of models.

Corollary 3.17. *For any $X \subseteq \mathsf{Te}_{TL}$, if X is t-inconsistent, then there exists no model $\mathfrak{M}_{TL}$ appropriate for X.*

Proof. Take any tableau inconsistent set of expressions X and any model $\mathfrak{M}_{\mathbf{TL}} = \langle D, d \rangle$. From the definition of tableau inconsistent set of expressions 3.14 it follows that:

1. there exists such formula $A \in \mathsf{For}_{\mathbf{TL}}$ that $A \in X$ and $\circ(A) \in X$,
   or
2. there exists such name letter $P \in \mathsf{Ln}$ and such number $i \in \mathbb{N}$ that $P_{+i} \in X$ and $P_{-i} \in X$.

If the first case occurs, then from fact 3.13 we know that $\mathfrak{M}_{\mathbf{TL}} \nvDash A$ or $\mathfrak{M}_{\mathbf{TL}} \nvDash \circ(A)$. If the second case occurs, then from definition of model 3.3 we know that there exists no such function $\gamma : \mathbb{N} \longrightarrow D$ that for each $j \in \mathbb{N}$:

1. if $P_{+j} \in X$, then $\gamma(j) \in d(P)$
2. if $P_{-j} \in X$, then $\gamma(j) \notin d(P)$.

Since then $\gamma(i) \in d(P)$ and $\gamma(i) \notin d(P)$. Hence, from the definition of model appropriate for the set of expressions 3.16 it follows that $\mathfrak{M}_{\mathbf{TL}}$ is not a model appropriate for set of expressions X. Whereas from the arbitrariness of model $\mathfrak{M}_{\mathbf{TL}}$ it follows that there does not exist model $\mathfrak{M}_{\mathbf{TL}}$ appropriate for X. □

3.3.1 Tableau rules for TL

The starting point for the construction of a tableau system for **TL** should be a precise definition of the concept of tableau rule. Before we proceed to the general concept of rule, we will introduce a certain auxiliary function $* : \mathsf{Te}_{\mathbf{TL}} \setminus \mathsf{For}_{\mathbf{TL}} \longrightarrow \mathbb{N}$ such that for any $P \in \mathsf{Ln}$ and any $i \in \mathbb{N}$:

- $*(P_{+i}) = i$
- $*(P_{-i}) = i$.

To each expression not being a formula, meaning a name letter with an index, function $*$ attributes an index which is found in it.

Similar to the case of **CPL**, we will first provide the general concept of rule. Not only because it allows to provide the general conditions that a tableau rule must meet. In the case of **TL** we will also provide alternative sets of tableau rules that are suitable for construction of a tableau system for **TL**. This means that within the below general concept of a tableau rule, we can define different sets of tableau rules that define various, however equivalent in terms of scope for correct inferences, tableau systems for **TL** (see note 3.20).

Definition 3.18 (Rule)**.** Let $P(\mathsf{Te}_{\mathbf{TL}})$ be a power set of the set of tableau expressions. Let $P(\mathsf{Te})^n$ be n–ry Cartesian product $\underbrace{P(\mathsf{Te}_{\mathbf{TL}}) \times \cdots \times P(\mathsf{Te}_{\mathbf{TL}})}_{n}$, for some $n \in \mathbb{N}$.

- By a *rule* we understand any subset $R \subseteq P(\mathsf{Te}_{\mathbf{TL}})^n$ such that if $\langle X_1, \ldots, X_n \rangle \in R$, then:
  1. X_1 is t-consistent
  2. $X_1 \subset X_i$, for each $1 < i \leq n$.
- If $n \geq 2$, then each element R will be called *ordered n-tuple* (pair, triple, etc., respectively).
- The first element of each n-tuples will be called an *input set* (*set of premises*), while its remaining elements *output sets* (*sets of conclusions*).

Definition of rule for **TL** differs from the definition of rule for **CPL** 2.12 among other things[4] in having introduced to the definition of rule for **TL** a condition of t-consistency of the input sets. We will no longer put down the rules in the form of sets, but immediately in the schematic, fractional form. Thus, from the general definition of rule itself, it follows that the input sets of tableau rules will be t-consistent.

A set of tableau rules designed for the defined tableau system for **TL** shall be introduced by means of the following definition.

Definition 3.19 (Tableau rules for **TL**)**.** *Tableau rules for* **TL** are the following rules:

$$R\mathbf{a}_+ : \frac{X \cup \{P\mathbf{a}Q, P_{+j}\}}{X \cup \{P\mathbf{a}Q, P_{+j}, Q_{+j}\}}$$

$$R\mathbf{e}_- : \frac{X \cup \{P\mathbf{e}Q, P_{+j}\}}{X \cup \{P\mathbf{e}Q, P_{+j}, Q_{-j}\}}$$

$$R\mathbf{i} : \frac{X \cup \{P\mathbf{i}Q\}}{X \cup \{P\mathbf{i}Q, P_{+j}, Q_{+j}\}}, \text{ where:}$$

1. $j \notin *(X \setminus \mathsf{For}_{\mathbf{TL}})$
2. for any $k \in \mathbb{N}$, $\{P_{+k}, Q_{+k}\} \nsubseteq X$.

$$R\mathbf{o} : \frac{X \cup \{P\mathbf{o}Q\}}{X \cup \{P\mathbf{o}Q, P_{+j}, Q_{-j}\}}, \text{ where:}$$

1. $j \notin *(X \setminus \mathsf{For}_{\mathbf{TL}})$
2. for any $k \in \mathbb{N}$, $\{P_{+k}, Q_{-k}\} \nsubseteq X$.

Set of tableau rules for **TL** will be defined as $\mathbf{R}_{\mathbf{TL}}$.

According to the general definition of rule 3.18, the input sets of each rule are t-consistent. In addition, in each tableau rule, the input set is basically contained in the output set. The notations provided are schemes of pairs belonging to the rules, so for each of the rules *X* is any set of expressions, *P*, *Q* are any name letters and *j* is any index such that all these elements satisfy the conditions imposed on the rule. One novelty is that set $\mathbf{R}_{\mathbf{TL}}$ contains rules that must include at least two premises, e.g. rule $R\mathbf{a}_+$.

4 We write *among other things* since this definition above all differs in the set on which we define rules. Nevertheless, all the other conditions are nearly identical to those of the concept of rule for **CPL**.

Another new and important aspect of the tableau rules in this system for **TL** are conditions in rules $R\mathbf{i}$, $R\mathbf{o}$. In both rules, we basically have the same conditions, so through one example we will discuss them collectively.

These conditions must be met if a given set is to be an output set with an assumed input set. So, the notation adopted in both rules says, for example, that if pair $\langle X \cup \{P\mathbf{i}Q\}, X \cup \{P\mathbf{i}Q, P_{+j}, Q_{+j}\}\rangle$ belongs to rule $R\mathbf{i}$, then:

1. $j \notin *(X \setminus \mathsf{For}_{\mathbf{TL}})$ and
2. for any $k \in \mathbb{N}$, $\{P_{+k}, Q_{+k}\} \nsubseteq X$.

These conditions are therefore necessary conditions for a given pair to belong to the rule.

Condition 1 requires the index which is entered to be new, meaning not appearing in any expression belonging to the output set. The semantic intuitions behind this procedure require the object denoted by new index to be new as well and not to remain in any positive relationship with the other names that appear in a given proof sequence.

In turn, condition 2 requires the input of a pair of expressions (in this example pair P_{+j}, Q_{+j}) to take place only when a similar pair does not already belong to the output set. In practice, this condition makes it impossible to enter unnecessary expressions in the proof, as is the case in example 3.21.

Analogous conditions will be considered in the next chapter which will be devoted to modal logic. There, we are going to provide an example which will illustrate the problem of infinite branches. In that case, even applying conditions blocking the unnecessary use of rules will not prevent the emergence of infinite branches. It will, however, prevent the creation of infinite branches in situations where this is not a consequence of logic itself, but of the wrong definition of the tableau system.

Remark 3.20. We can consider alternative sets of rules for the construction of a tableau system for **TL**. The following rule would help.

$$R\mathbf{e}'_{-} : \frac{X \cup \{P\mathbf{e}Q, Q_{+j}\}}{X \cup \{P\mathbf{e}Q, Q_{+j}, P_{-j}\}}$$

Rule $R\mathbf{e}'_{-}$ allows to proceed from premises $P\mathbf{e}Q$, Q_{+j} to conclusion P_{-j}. The semantic intuition contained in this rule says that if a name from the predicate in a general contradicting proposition has a subject j as the designator, then this object is not the designator of the subject of this proposition.

Making use of rule $R\mathbf{e}'_{-}$, we can define the following sets of tableau rules, different from set $\mathbf{R}_{\mathbf{TL}}$:

(a) $\mathbf{R_{TL}} \cup \{R\mathbf{e}'_{-}\}$

(b) $(\mathbf{R_{TL}} \setminus \{R\mathbf{e}_{-}\}) \cup \{R\mathbf{e}'_{-}\}$.

Likewise, we could consider another rule that says that if we have a proposition $P\mathbf{a}Q$ and expression Q_{-j} saying — intuitively — that an object denoted by j is not a designator of name Q, then this object is not a designator of name P either.

$$R\mathbf{a}_{-} : \frac{X \cup \{P\mathbf{a}Q, Q_{-j}\}}{X \cup \{P\mathbf{a}Q, Q_{-j}, P_{-j}\}}$$

Making use of rule $R\mathbf{a}_{-}$, we can define the following sets of tableau rules, different from set $\mathbf{R_{TL}}$:

(a) $\mathbf{R_{TL}} \cup \{R\mathbf{a}_{-}\}$

(b) $(\mathbf{R_{TL}} \setminus \{R\mathbf{a}_{+}\}) \cup \{R\mathbf{a}_{-}\}$.

Furthermore, extending the language of logic of **TL** to the language which includes expressions denoting statements that a particular object is or is not the designator of a given name, we could consider another rule that allows "reversing" general contradicting propositions:

$$R\mathbf{e} : \frac{X \cup \{P\mathbf{e}Q\}}{X \cup \{Q\mathbf{e}P\}}$$

It is worth noting that the tableau system for **TL** we are now describing, there are no branching rules (all tableau rules are sets of pairs). So there will be no branchings in the tableaux. However, we might consider such variant of rule $R\mathbf{a}_{+}$ which would allow branchings:

$$R\mathbf{a}'_{+} : \frac{X \cup \{P\mathbf{a}Q\}}{X \cup \{P\mathbf{a}Q, P_{-i}\} | X \cup \{P\mathbf{a}Q, Q_{+i}\}}, \text{ where } i \in *(X)$$

However, due to the economy of tableau proof, it is better — as far as possible — to introduce the fewest possible number of tableau rules that are not sets of pairs.

Although all these rules and sets of rules seem to be interesting, we will not consider them all — neither for the language of **TL**, nor for extended languages — for the reasons described in the previous chapter (note 2.19). Instead, we will focus on the tableau system determined by set of rules $\mathbf{R_{TL}}$.

Nonetheless, let us stress once again that the examples provided indicate that the general definition of rule 3.18 makes sense. There may exist many sets of tableau rules that potentially — we mean (potentially) because to determine this it always requires a proof — they define the same logic.

Example 3.21. Starting from set $X \cup \{P\mathbf{i}Q\}$ and using rule $R\mathbf{i}$ without condition 2, we could get an infinite sequence in which we enter a new index in each of the elements:

$$\begin{array}{c} R\mathrm{i}\ X_1 = X \cup \{P\mathrm{i}Q\} \\ | \\ R\mathrm{i}\ X_2 = X_1 \cup \{P_{+\mathrm{i}}, Q_{+\mathrm{i}}\} \\ | \\ R\mathrm{i}\ X_3 = X_2 \cup \{P_{+\mathrm{j}}, Q_{+\mathrm{j}}\} \\ | \\ R\mathrm{i}\ X_4 = X_3 \cup \{P_{+\mathrm{k}}, Q_{+\mathrm{k}}\} \\ | \\ \ldots \end{array}$$

Such a sequence is infinite not because of the properties of the logic itself, but because of the unnecessary acceptance of continuous application to the same elements of the rule already applied once.

The adopted set of tableau rules — in our case set $\mathbf{R_{TL}}$ — determines the content of the range of successive concepts of the tableau system. Although the formal concepts that we will describe will be analogous to those from the previous chapter, each of them will depend on set $\mathbf{R_{TL}}$.

3.3.2 Branches for TL

With a fixed set of tableau rules, we can proceed to the concept of branch. As we already know, branches are such sequences of sets that each two adjacent elements are in turn: an input set and an output set of some n-tuple that belongs to the set of tableau rules. Branches are therefore setwise objects consisting of sets. Below, we present the formal definition of branch in the tableau system for **TL**.

Definition 3.22 (Branch)**.** Let $K = \mathbb{N}$ or $K = \{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$. Let X be any set of expressions. *A branch* (or *a branch beginning with* X) will be called any sequence $\phi : K \longrightarrow P(\mathsf{Te}_{\mathbf{TL}})$ that meets the following conditions:

1. $\phi(1) = X$
2. for any $i \in K$: if $i + 1 \in K$, then there exists such rule $R \in \mathbf{R_{TL}}$ and such pair $\langle Y_1, Y_2 \rangle \in R$ that $\phi(X_i) = Y_1$ i $\phi(X_{i+1}) = Y_2$.

Having two branches ϕ, ψ such that $\phi \subset \psi$ we shall state that:

- ϕ is a sub-branch of ψ
- ψ is a super-branch of ϕ.

Denotation 3.23. From now on — when speaking of branches for **TL** — for convenience, we will use the following notations or designations:

1. $X_1, \ldots, X_n$, where $n \geq 1$
2. $\langle X_1, \ldots, X_n \rangle$, where $n \geq 1$
3. abbreviations: ϕ_M (where M is a domain ϕ, i.e. $\phi : M \longrightarrow P(\mathsf{Te}_{\mathbf{TL}}))$
4. or — to denote branches — small Greek letters: ϕ, ψ, etc.

The sets of branches, in turn, we shall denote with capital Greek letters: Φ, Ψ, etc. Furthermore, the domain cardinality of a given branch K we shall sometimes call a *length* of that branch.

Remark 3.24. We will repeat here in part the remark from the previous chapter. As we can see, the concept of branch depends on some set of tableau rules. In the case under consideration, the branch structure is based on the rules from set $\mathbf{R_{TL}}$. Further described complex tableau concepts will also depend on some sets of rules. Because in this chapter we are studying tableau system for **TL** based on rules from set $\mathbf{R_{TL}}$, so we are not going to make it any more complicated.

In practice, however, the tableau concepts of systems constructed according to the presented idea always base on some set of rules. In one of the subsequent chapters, at the general description of the construction way itself, the set of rules will be a variable. In this chapter it is specified as: $\mathbf{R_{TL}}$ and the complex tableau concepts defined here depend on it. And since set $\mathbf{R_{TL}}$ only includes such rules that constitute sets of ordered pairs, so in the definition of branch for **TL** we specified that it is about pairs, contrary to the definition of tableau system in the previous chapter, where in the definition of branch 2.20, we wrote about the existence of an appropriate n-tuple.

This remark applies to the rest of the study.

By definition of rule 3.18, through the fact that the rules are defined by proper containing of the input set in each of the output sets, in any n-tuple, there is a conclusion.

Corollary 3.25. *Each branch is an injective sequence.*

3.3.3 Maximal branches

Among the branches constructed through applying the rules from set $\mathbf{R_{TL}}$, we will distinguish such branches to which no more rules from set $\mathbf{R_{TL}}$ can be applied, expanding them into some super-branches. As we already know, such branches are called maximal branches. The definition of maximal branch is the same as in the previous chapter, except that of course the maximal branches here are branches for **TL**.

Definition 3.26 (Maximal branch). Let $\phi : K \longrightarrow P(\mathsf{Te}_{\mathbf{TL}})$ be a branch. We shall state that ϕ is *maximal* iff

1. $K = \{1, 2, 3, \ldots, n\}$, for some $n \in \mathbb{N}$
2. there is no branch ψ such that $\phi \subset \psi$.

As in the case described in the previous chapter, definition 3.26 could be shortened to the second condition. This would not change anything with regard to the cases of finite sets, but we would allow cases of infinite branches that:

- begin with an infinite set
- meet the second condition of definition because they are infinite branches
- do not, however, "resolve" all expressions.

The following example 3.27 illustrates a situation where without the first condition in the definition of maximal branch 3.26, we would consider a presented branch to be maximal. This branch would be infinite and, at the same time, would still contain expressions that were not used in the proof.

Example 3.27. Let us consider the following branch, beginning from set $\{P^1 \mathbf{i} Q^1\} \cup \{P^n \mathbf{i} Q^n : n > 1\}$.

$$
\begin{array}{rcl}
R\mathrm{i}\ X_1 & = & \{P^1 \mathrm{i} Q^1\} \cup \{P^n \mathrm{i} Q^n : n > 1\} \\
& | & \\
R\mathrm{i}\ X_2 & = & X_1 \cup \{P^2_{+1}, Q^2_{+1}\} \\
& | & \\
R\mathrm{i}\ X_3 & = & X_2 \cup \{P^3_{+2}, Q^3_{+2}\} \\
& | & \\
R\mathrm{i}\ X_4 & = & X_3 \cup \{P^4_{+3}, Q^4_{+3}\} \\
& | & \\
& \ldots &
\end{array}
$$

This branch is infinite, although rule $R\mathbf{i}$ has never been applied to any of the sets in such a way as to draw conclusions from formula $P^1 \mathbf{i} Q^1$.

Thus, the definition of maximal branch 3.26 says that if sequence $X_1, \ldots, X_n$, for some $n \geq 1$, is a branch, then we call it a *maximal* branch iff there does not exist branch $X_1, \ldots, X_n, X_{n+1}$.

Extending the concept of branch onto infinite sets is unnecessary in practice, because important concepts for our metatheory — the concept of branch consequence and tableau — we apply to cases of finite sets. At this stage of the book, the finite sets of tableau expressions do not begin the branch of infinite length.

However, the definition of maximal branch will change in the study. To the above concept of branch applicable are all reservations from note 2.32, concerning the maximal branch for the tableau system for **CPL**. Definition 3.26 is suitable for those tableau systems where only finite branches are obtained from finite sets of expressions. For other systems this definition is too narrow. It does not include cases of branches which, even though they begin with a finite set of expressions, are not finite.

In the following chapter, we will proceed to defining the system for modal logic, and we will generalize this definition. Hence, the above definition 3.26, and especially definition 2.20 will describe special cases of maximal branches which appear in the construction by the described method of such tableau systems as system for **CPL** or **TL**. Besides, also in the case of **TL** we will show that from finite sets of expressions we always get branches of finite length (fact 3.32).

3.3.4 Closed and open branches

Among the maximal branches, the closed branches deserve special attention. In addition, set of open branches complements the set of closed branches. As we remember, intuitively a branch is closed when we get a t-inconsistent set having decomposed tableau expressions.

Definition 3.28 (Closed/open branch)**.** A branch $\phi : K \longrightarrow P(\mathsf{Te}_{\mathbf{TL}})$ will be called *closed* iff $\phi(i)$ is a t-inconsistent set, for some $i \in K$. A branch will be called *open* iff it is not closed.

From the above definition, the definition of tableau rules for **TL** 3.19 and the definition of branch 3.22 the following conclusion follows.

Corollary 3.29. *If branch $\phi : K \longrightarrow P(\mathsf{Te}_{\mathbf{TL}})$ is closed, then $|K| \in \mathbb{N}$ and $\phi(|K|)$ is a t-inconsistent set.*

In the case of a closed branch, the t-inconsistent sequence element is the last element because no rule can be applied to it anymore to extend the branch. For the rules are defined in such a way that they cannot be applied to t-inconsistent sets. Therefore, from the definition of maximal 3.26 another conclusion follows.

Corollary 3.30. *If branch $f : K \longrightarrow P(\mathsf{Te}_{\mathbf{TL}})$ is closed, then it is maximal.*

We are now going to show two facts that are needed for further proofs. The first one says that a branch that begins with a finite set of expressions is also finite in length, not greater than a certain number.

Proposition 3.31. *Let X be a finite set of expressions. Let $\phi : K \longrightarrow P(\mathsf{Te}_{\mathbf{TL}})$ be any such branch that $\phi(1) = X$. Then, there exists number $n \in \mathbb{N}$ such that $|K| \leq n$.*

Proof. Take any finite set of expressions X and such branch $\phi : K \rightarrow P(\mathsf{Te}_{\mathbf{TL}})$ that $\phi(1) = X$. We will carry out an inductive proof due to the cardinality of the first element of the branch.

Initial step. Assume that $|X| = 1$. We have six types of cases that can take place. There exist name letters P, $Q \in \mathsf{Ln}$ and index $i \in \mathbb{N}$ such that one of the following cases occurs:

1. $P_{+i} \in X$, then, however, $|K| = 1 \in \mathbb{N}$, by definition of branch 3.22, since there does not exist such rule $R \in \mathbf{R_{TL}}$ that would allow to extend branch $\langle X \rangle$
2. $P_{-i} \in X$, then, however, $|K| = 1 \in \mathbb{N}$, by definition of branch 3.22, since there does not exist such rule $R \in \mathbf{R_{TL}}$ that would allow to extend branch $\langle X \rangle$
3. $P\mathbf{a}Q \in X$, then, however, $|K| = 1 \in \mathbb{N}$, by definition of branch 3.22, since there does not exist such rule $R \in \mathbf{R_{TL}}$ that would allow to extend branch $\langle X \rangle$
4. $P\mathbf{i}Q \in X$, then, however, $|K| \leq 2 \in \mathbb{N}$, by definition of branch 3.22, since there only exists one rule $R \in \mathbf{R_{TL}}$, rule $R\mathbf{i}$ that allows to extend branch $\langle X \rangle$ with set $Y = \{P\mathbf{i}Q, P_{+j}, Q_{+j}\}$, for some $j \in \mathbb{N}$, whereas there does not exist such rule $R \in \mathbf{R_{TL}}$ that would allow to extend branch $\langle X, Y \rangle$
5. $P\mathbf{e}Q \in X$, then, however, $|K| = 1 \in \mathbb{N}$, by definition of branch 3.22, since there does not exist such rule $R \in \mathbf{R_{TL}}$ that would allow to extend branch $\langle X \rangle$
6. $P\mathbf{o}Q \in X$, then, however, $|K| \leq 2 \in \mathbb{N}$, by definition of branch 3.22, since there only exists one rule $R \in \mathbf{R_{TL}}$, rule $R\mathbf{o}$ that allows to extend branch $\langle X \rangle$ with set $Y = \{P\mathbf{i}Q, P_{+j}, Q_{+j}\}$, for some $j \in \mathbb{N}$, whereas there does not exist such rule $R \in \mathbf{R_{TL}}$ that would allow to extend branch $\langle X, Y \rangle$.

Thus, when $|X| = 1$, then $|K| \leq 2$. So, if $|X| = 1$, then there exists such number $n \in \mathbb{N}$ that $|K| \leq n$.

Induction step. Assume that the theorem thesis holds for each such set of expressions Y that $|Y| = m$. Thus, for any branch $\psi : M \longrightarrow P(\mathsf{Te}_{\mathbf{TL}})$ such that $\psi(1) = Y$, there exists number l such that $|M| \leq l$.

We will show that the theorem thesis also occurs for $|X| = m+1$. Take any set of expressions Y such that $Y \subseteq X$ and $|Y| = m$. Thus, for any branch $\chi : N \longrightarrow P(\mathsf{Te}_{\mathbf{TL}})$ such that $\chi(1) = Y$ there exists number l such that $|N| \leq l$.

We have six types of cases that can take place. There exist name letters P, Q $\in \mathsf{Ln}$ and index $i \in \mathbb{N}$ such that one of the following cases occurs:

1. $X = Y \cup \{P_{+i}\}$, then, however, $|K| \leq l + k \in \mathbb{N}$, where k is the number of propositions in form $R\mathbf{a}S$ and $T\mathbf{e}U$ that belong to set Y, by definition of branch 3.22, since set $\mathbf{R_{TL}}$ contains rules $R\mathbf{a}_{+}$ and $R\mathbf{e}_{-}$ which allow to extend each branch containing additional expression P_{+i} at most with k elements
2. $X = Y \cup \{P_{-i}\}$, then, however, $|K| \leq l \in \mathbb{N}$, by definition of branch 3.22, since set $\mathbf{R_{TL}}$ does not comprise any rule which would allow to extend a branch containing additional expression P_{-i}
3. $X = Y \cup \{P\mathbf{a}Q\}$, then, however, $|K| \leq l + ((k+1) \cdot o) \in \mathbb{N}$, where k is the number of propositions in form $R\mathbf{a}S$ and $T\mathbf{e}U$ that belong to set Y, while o is the number

of particular propositions that belong to set Y and in the subject or predicate have name letter P, and expressions P_{+j}, for some index j, by definition of branch 3.22, since set $\mathbf{R_{TL}}$ contains rules $R\mathbf{a}_+$ and $R\mathbf{e}_-$ which allow to extend each branch containing additional expression $P\mathbf{a}Q$ at most with $(k+1)\cdot o$ elements

4. $X = Y \cup \{P\mathbf{i}Q\}$, then, however, $|K| \leq l + k + 1 \in \mathbb{N}$, where k is the number of propositions in form $R\mathbf{a}S$ and $T\mathbf{e}U$ that belong to set Y, by definition of branch 3.22, since set $\mathbf{R_{TL}}$ contains rules $R\mathbf{i}$, $R\mathbf{a}_+$ and $R\mathbf{e}_-$, which allow to extend each branch containing additional expression $P\mathbf{i}Q$ at most with $k+1$ elements
5. $X = Y \cup \{P\mathbf{e}Q\}$, then, however, $|K| \leq l + k \in \mathbb{N}$, where k is the number of particular propositions that belong to set Y and in the subject or predicate have name letter P, and expressions P_{+j}, for some index j, by definition of branch 3.22, since set $\mathbf{R_{TL}}$ contains rule $R\mathbf{e}_-$, which allows to extend each branch containing additional expression $P\mathbf{e}Q$ at most with k elements
6. $X = Y \cup \{P\mathbf{o}Q\}$, then, however, $|K| \leq l + k + 1 \in \mathbb{N}$, where k is the number of propositions in form $R\mathbf{a}S$ and $T\mathbf{e}U$ that belong to set Y, by definition of branch 3.22, since set $\mathbf{R_{TL}}$ contains rules $R\mathbf{o}$, $R\mathbf{a}_+$ and $R\mathbf{e}_-$, which allow to extend each branch containing additional expression $P\mathbf{o}Q$ at most with $k+1$ elements.

So, if $|X| = m+1$, then there exists such number $n \in \mathbb{N}$ that $|K| \leq n$. Then, there exists such number $n \in \mathbb{N}$ that $|K| \leq n$. □

The second fact says that for each finite set of expressions, there exists a maximal branch which begins with this set.

Proposition 3.32. *Let $X \subseteq \mathsf{Te}_{\mathbf{TL}}$ be a finite set of expressions. Then, there exists maximal branch $\phi : K \longrightarrow P(\mathsf{Te}_{\mathbf{TL}})$ such that $\phi(1) = X$.*

Proof. Let $X \subseteq \mathsf{Te}_{\mathbf{TL}}$ be a finite set of expressions. Then, by fact 3.31, for each branch $\phi : K \longrightarrow P(\mathsf{Te}_{\mathbf{TL}})$ such that $\phi(1) = X$, there exists such number $n \in \mathbb{N}$ that $|K| \leq n$.

Indirectly assume that no branch ϕ beginning with set X is a maximal branch. By an inductive proof through the branch length we will show that this assumption leads to a contradiction.

Initial step. We know there exists at least one branch. This is the branch that begins with set X and that a length of 1. However, by indirect assumption, it is not a maximal branch, so by definition of maximal branch 3.26, there exists a branch that begins with X, with length of 2.

Induction step. Now, let us take a branch beginning with X, of length m, and assume there exists a branch of length $m+1$ beginning with set X. Again, by indirect assumption, it is not a maximal branch, so by definition of maximal branch 3.26, there exists a branch that begins with X of length $m+2$.

Thus for any $m \in \mathbb{N}$, there exists a super-branch of length $m+1$ beginning with finite set of expressions X. This, however, contradicts the fact that for any branch beginning with set X there exists such number $n \in \mathbb{N}$ that bounds the length of that branch. □

3.3.5 Relation of branch consequence

We will now define the concept of branch consequence using the concepts of branch, maximal branch and closed branch.

Definition 3.33 (Branch consequence of **TL**). Let $X \subseteq \mathsf{For}_{\mathbf{TL}}$ and $A \in \mathsf{For}_{\mathbf{TL}}$. Formula A is a *branch consequence* of X (for short: $X \triangleright A$) iff there exists such finite set $Y \subseteq X$ that each maximal branch beginning with set $Y \cup \{\circ(A)\}$ is closed.

Denotation 3.34. For any set of formulas X and any formula A notation $X \not\triangleright A$ means that it is not the case that $X \triangleright A$.

In the definition of branch consequence, we refer to the function specified by definition 3.12. Function $\circ$ to each formula assigns a formula which contradicts it 3.13. The above concept of branch consequence differs from the analogous concept for **CPL** in the fact that (apart from defining on a different language — set of expressions) in the previous case negation was used, and here we use a contradictory formula. In fact, however, it can be assumed that in both cases the point is to start with a formula contradictory to the formula which could potentially be a branch consequence of a certain set of premises, but in the case of **CPL** a contradictory formula could be obtained directly by preceding the formula with a negation.

Also in the present case, we will refer to the example of the branch consequence occurrence.

Example 3.35. Let us consider an example — Barbara syllogism — premises $\{P\mathbf{a}Q, Q\mathbf{a}R\}$, conclusion $P\mathbf{a}R$[5]. We want to answer the question whether $\{P\mathbf{a}Q, Q\mathbf{a}R\} \triangleright P\mathbf{a}R$?

The first set of each branch we need to examine is set $\{P\mathbf{a}Q, Q\mathbf{a}R, \circ(P\mathbf{a}R)\}$, by definition of function $\circ$, equal to set $\{P\mathbf{a}Q,\ Q\mathbf{a}R,\ P\mathbf{o}R\}$. On the left side, we put the branch elements, on the right side, we put the rules we use to transform the sets. There exists one type of maximal branches beginning with this last set:

1. $\{P\mathbf{a}Q, Q\mathbf{a}R, P\mathbf{o}R\} \subset$ $R\mathbf{o}$, where $i \in \mathbb{N}$

5 Note that we are considering a pattern of inference rather than a specific inference as we use metavariables and letters instead of indices. It is obvious, however, that further considerations would be identical if we used name letters and digits.

2. $\{P\mathbf{a}Q, Q\mathbf{a}R, P\mathbf{o}R, P_{+i}, R_{-i}\} \subset$ $R\mathbf{a}_{+}$
3. $\{P\mathbf{a}Q, Q\mathbf{a}R, P\mathbf{o}R, P_{+i}, R_{-i}, Q_{+i}\} \subset$ $R\mathbf{a}_{+}$
4. $\{P\mathbf{a}Q, Q\mathbf{a}R, P\mathbf{o}R, P_{+i}, R_{-i}, Q_{+i}, R_{+i}\}$ t-inconsistency R_{-i}, R_{+i}

We said one *type* since i is any natural number. So, actually there exists an infinite number of branches beginning with set $\{P\mathbf{a}Q, Q\mathbf{a}R, P\mathbf{o}R\}$.[6] The described branches are maximal ones because their last element is t-inconsistent as it contains expressions R_{-i}, R_{+i}, always for some $i \in \mathbb{N}$.

So we showed that all maximal branches beginning with set $\{P\mathbf{a}Q, Q\mathbf{a}R, \circ(P\mathbf{a}R)\}$ are closed. Thus $\{P\mathbf{a}Q, Q\mathbf{a}R\} \rhd P\mathbf{a}R$.

The tableau concepts we have described for **TL** are very similar to their equivalents for **CPL**, despite the fact that we have dealt with a different language of proof and with indices.

3.4 Tableaux for TL

As in the case of the tableau system for **CPL**, the practical examining of the branch consequence occurrence should boil down to building of an appropriate tableau.

Nevertheless, we are going to have a discussion on the definition of the concepts of tableau, complete tableau, closed/open tableau of a tableau system for **TL**. We will define all these concepts in two variants.

The first variant will make a direct reference to the fact that among the rules belonging to set $\mathbf{R_{TL}}$ there are no rules that contain an ordered n-tuples longer than two, and as a consequence there never occur any branchings. The second variant of the concepts will be based on the relevant definitions already provided in the tableau system for **CPL**, however related to the set of branches of the tableau system for **TL**.

Although in each case the first definition variant will appear to be a special and simplified case of the second variant, in fact the two variants will prove to be equivalent, as we will demonstrate. In this way we will also show that the book considerations aim to describe certain general concepts for the tableau systems, independent of many specific properties of a given system, and thus to the general theory of tableau systems.

6 Of course, we might define equivalents of rules $R\mathbf{i}$ and $R\mathbf{o}$ in such a way that the introduced expressions as indices only had for instance the least number that does not appear in the set to which we apply the rule. Then, the number of branches would amount to one. There are many ways to define these rules similarly, but from the formal point of view each time we would then examine a different axiomatization than the assumed set of rules $\mathbf{R_{TL}}$.

Therefore, the first variant of each definition will always be a specific version, defined according to the needs of the current system, while the second one will be a definition analogous to the one used in the previous chapter. As usual, we will start with the definition of tableau.

Definition 3.36 (Tableau — variant one)**.** Let $X \subseteq \mathsf{For}_{\mathbf{TL}}$, $A \in \mathsf{For}_{\mathbf{TL}}$ and Φ be a set of branches. Ordered triple $\langle X, A, \Phi \rangle$ will be called a *tableau for* $\langle X, A \rangle$ (or for short *tableau*) iff Φ is an one-element subset of the set of branches beginning with set $X \cup \{\circ(A)\}$ (i.e. if $\phi \in \Phi$, then $\phi(1) = X \cup \{\circ(A)\}$).

Remark 3.37. According to the definition of tableau 3.36, a tableau for pair $\langle X, A \rangle$ is ordered triple $\langle X, A, \Phi \rangle$ in which Φ is such an one-element set of branches that any branch in this set begins with set $X \cup \{\circ(A)\}$. This definition could be technically simplified.

Such simplification would consist in the fact that in an ordered triple, instead of set of branches Φ, we would simply place a branch belonging to set Φ. However, as we strive for a general theory of tableaux, so we use a general notation of a tableau as a triple in which we distinguish a set of premises, a potential conclusion and a set of branches that meet the conditions from the definition. The case of the system described in the present chapter is in terms of the number of branches — as we wrote at the beginning of the chapter — a borderline case, so we will not alter the convention for it.

Let us now define the concept of tableau for **TL** in a similar way the concept of tableau for **CPL**. Similar, since this definition is analogous to the corresponding definition for the tableau system for **CPL**. However, we cannot say that it is identical, because it pertains to the objects that were constructed from other sets, in spite of similarity of the same objects, i.e. branches. This definition uses — just like the definition of tableau 2.53 in the tableau system for **CPL** — the concept of maximal branch in the set of branches which we will adapt to the current situation. Here, the analogous remarks apply as in the case of definition 2.36 from Chapter Two — so we will not reiterate those here.

Definition 3.38 (Maximal branch in the set of branches)**.** Let Φ be a set of branches and let branch $\psi \in \Phi$. Branch ψ will be called *maximal in* Φ (or Φ–*maximal*) iff there is no such branch $\phi \in \Phi$ that $\psi \subset \phi$.

With the concept of maximal branch in the set of branches, we can proceed to the second variant of the tableau definition.

Definition 3.39 (Tableau — variant two)**.** Let $X \subseteq \mathsf{For}_{\mathbf{TL}}$, $A \in \mathsf{For}_{\mathbf{TL}}$ and Φ be a set of branches. Ordered triple $\langle X, A, \Phi \rangle$ will be called *a tableau for* $\langle X, A \rangle$ (or for short *tableau*) iff the below conditions are met:

1. Φ is a non-empty subset of set of branches beginning with $X \cup \{\circ(A)\}$ (i.e. if $\phi \in \Phi$, then $\phi(1) = X \cup \{\circ(A)\}$)
2. each branch contained in Φ is Φ-maximal
3. for any $n, i \in \mathbb{N}$ and any branches $\psi_1, \ldots, \psi_n \in \Phi$, if:
   - i and $i+1$ belong to domains of functions $\psi_1, \ldots, \psi_n$
   - for any $1 < k \leq n$ and any $o \leq i$, $\psi_1(o) = \psi_k(o)$

   then there exists such rule $R \in \mathbf{R_{TL}}$ and such ordered m-tuple $\langle Y_1, \ldots, Y_m \rangle \in R$, where $1 < m \leq 3$ that for any $1 \leq k \leq n$:
   - $\psi_k(i) = Y_1$
   - and there exists such $1 < l \leq m$ that $\psi_k(i+1) = Y_l$.

The above tableau definition for **TL** seems too complex. It is not too broad because — as we will see — it does not cover setwise constructions that we would not consider as tableaux for the tableau system for **TL**. However, condition 3 of this definition seems to be overly expanded. Note the conclusion that follows from the definition of tableau in variant two and the definition of set of tableau rules $\mathbf{R_{TL}}$.

Corollary 3.40. *Let $\langle X, A, \Phi \rangle$ be a tableau in variant two 3.39. Then, set Φ contains precisely one branch.*

Proof. Let $\langle X, A, \Phi \rangle$ be a tableau. In the proof, we will make use of definition of tableau 3.39. Assume that set Φ contains two branches ϕ_1 and ϕ_2. Point 1. of the tableau definition implies that $\phi_1(1) = \phi_2(1) = X \cup \{\circ(A)\}$. While point 2. and definition of maximal branch in the set of branches 3.38 imply that $\phi_1 \notin \phi_2$ and $\phi_2 \notin \phi_1$ since each branch in set Φ is Φ-maximal.

Now, take any such number $i \in \mathbb{N}$ that for any $o \leq i$, $\phi_1(o) = \phi_2(o)$, plus $i+1$ belong to domains of functions ϕ_1 and ϕ_2. From point 3. it follows that there exists such rule $R \in \mathbf{R_{TL}}$ and such ordered m-tuple $\langle Y_1, \ldots, Y_m \rangle \in R$, where $1 < m \leq 3$ that for any $1 \leq k \leq 2$:

- $\phi_k(i) = Y_1$
- there exists such $1 < l \leq m$ that $\phi_k(i+1) = Y_l$.

And yet, since for each rule $R \in \mathbf{R_{TL}}$, for each ordered m-tuple $\langle Y_1, \ldots, Y_m \rangle \in R$, where $1 < m \leq 3$, $m = 2$, so $\phi_1(i+1) = Y_l = \phi_2(i+1)$. Hence $\phi_1 = \phi_2$. □

This conclusion shows that the concept of tableau in the second variant for **TL** system could be simpler. Nonetheless, it is not too broad since both definitions of tableau for the tableau system for **TL** are equivalent, which gets proven by the below fact.

Proposition 3.41. *Let $X \subseteq \mathsf{For}_{\mathbf{TL}}$, $A \in \mathsf{For}_{\mathbf{TL}}$ and Φ be a set of branches. $\langle X, A, \Phi \rangle$ is a tableau in variant one 3.36 iff $\langle X, A, \Phi \rangle$ is a tableau in variant two 3.39.*

Proof. Let $X \subseteq \mathsf{For}_{\mathbf{TL}}$, $A \in \mathsf{For}_{\mathbf{TL}}$ and Φ be a set of branches.

Assume that $\langle X, A, \Phi \rangle$ is a tableau in variant one 3.36. In view of this definition, Φ is a non-empty subset of set of branches beginning with $X \cup \{\circ(A)\}$ (i.e. if $\phi \in \Phi$, then $\phi(1) = X \cup \{\circ(A)\}$), so the first condition from definition 3.39 is satisfied. Definiens of definition 3.36 also claims that Φ contains precisely one branch. So, each branch that belongs to Φ is Φ-maximal by virtue of definition 3.38, which is the second condition of definition 3.39. Finally, also the third condition of definition 3.39 is met as Φ contains precisely one branch ψ. Thus, $\langle X, A, \Phi \rangle$ is a tableau in variant two.

Now, assume that $\langle X, A, \Phi \rangle$ is a tableau in variant two 3.39. Due to the first condition of this definition, Φ is a non-empty subset of set of branches beginning with $X \cup \{\circ(A)\}$ (i.e. if $\phi \in \Phi$, then $\phi(1) = X \cup \{\circ(A)\}$). While due to conclusion 3.40, Φ contains precisely one branch. Thus, as per definition 3.36 $\langle X, A, \Phi \rangle$ is a tableau in variant one. □

Therefore, any of the definitions of tableau in the tableau system considered here can be adopted for **TL**, although due to the economy of phrasing, definition 3.36 seems better. Its scope covers exactly the same objects as in the definition of tableau that was created by applying analogies to the definition of tableau in the system for **CPL**.

We will now take up the issue of the complete tableau, also considering simpler and more complex variants, modelled on the system for **CPL**. Here is variant one.

Definition 3.42 (Complete tableau — variant one)**.** Let triple $\langle X, A, \Phi \rangle$ be a tableau. We shall state that $\langle X, A, \Phi \rangle$ is *complete* iff a branch contained in Φ is maximal. A tableau is *incomplete* iff it is not complete.

In a complete tableau, a branch that belongs to it is maximal, so it cannot be extended. In variant two, which in a moment will be adapted to the context of **TL**, in a complete tableau all branches are maximal, not only those maximal ones in a given set. In addition, a complete tableau contains such set of branches that it is no longer possible to add any new branches to it, without causing the set to cease to be a tableau. More specifically, a complete tableau contains such set of maximal branches that any non-redundant superset of it ceases to co-create the tableau, similar to definition 2.60.

Consideration of the second variant, based on the tableau definition from the previous chapter, requires the introduction of an auxiliary definition, including the definition of a redundant superset.

Definition 3.43 (Redundant variant of branch)**.** Let ϕ and ϕ' be such branches that for some numbers i and $i+1$ that belong to their domains, it is the case that for any $j \leq i$, $\phi(j) = \phi'(j)$, but $\phi(i+1) \neq \phi'(i+1)$. We shall state that branch ϕ' is an *redundant variant* of branch ϕ iff:

- there exists such rule $R \in \mathbf{R_{TL}}$ and such pair $\langle X, Y \rangle \in R$ that $X = \phi(i)$ and $Y = \phi(i+1)$
- there exists such rule $R \in \mathbf{R_{TL}}$ and such triple $\langle X, W, Z \rangle \in R$ that $X = \phi(i)$ and:
  1. $W = \phi(i+1)$ and $Z = \phi'(i+1)$
     or
  2. $Z = \phi(i+1)$ and $W = \phi'(i+1)$.

Let Φ, Ψ be sets of branches and $\Phi \subset \Psi$. We shall state that Ψ is an *redundant superset* Φ iff for any branch $\psi \in \Psi \setminus \Phi$ there exists such branch $\phi \in \Phi$ that ψ is a redundant variant of ϕ.

However, in the case of our system, the concept of redundant variant of branch is empty. It is covered by another conclusion.

Corollary 3.44. *For no branch ϕ there exists such branch ψ that ψ is a redundant variant of branch ϕ.*

Proof. Let us consider any branch ϕ and indirectly assume that there exists such branch ψ that ψ is a redundant variant of branch ϕ. According to definition of redundant variant of branch 3.43, for some numbers i and $i+1$ that belong to domains ϕ and ψ, it is the case that $\phi(i) = \psi(i)$, but $\phi(i+1) \neq \psi(i+1)$, and there exists such rule $R \in \mathbf{R_{TL}}$ and such triple $\langle X, Y, Z \rangle \in R$ that $X = \phi(i)$ and $Y = \phi(i+1)$, $Z = \psi(i+1)$ or $Y = \psi(i+1)$, $Z = \phi(i+1)$. However, this leads to a contradiction to the fact that there exists no rule $R \in \mathbf{R_{TL}}$ to comprise any triple $\langle X, Y, Z \rangle$. □

And since there exist no redundant variants of branches, there are neither redundant nor proper supersets of sets of branches, which is confirmed by another conclusion.

Corollary 3.45. *Let Φ be a set of branches. There exists no set of branches Ψ such that $\Phi \subset \Psi$ and Ψ is a redundant superset of Φ.*

Proof. Take any set of branches Φ and indirectly assume that there exists set of branches Ψ such that $\Phi \subset \Psi$ and Ψ is a redundant superset of Φ. Thus — according to definition of redundant variant of branch 3.43 — for any branch $\psi \in \Psi \setminus \Phi$, there exists such branch $\phi \in \Phi$ that ψ is a redundant variant of ϕ. Since $\Phi \subset \Psi$, so there exists some branch $\psi \in \Psi \setminus \Phi$. Assume that set Φ is non-empty. Consequently, there exists such branch $\phi \in \Phi$ that some ψ is a redundant variant of ϕ, which contradicts the previous conclusion 3.44. □

We have shown that both concepts — of a redundant variant of branch and a redundant superset, in the phrasing appropriate for our tableau system for **TL**, are empty. After considering the redundant variants of branches and redundant supersets of branches, we can proceed to the definition of complete tableau in variant two.

Definition 3.46 (Complete tableau — variant two)**.** Let $\langle X, A, \Phi\rangle$ be a tableau. We shall state that $\langle X, A, \Phi\rangle$ is *complete* iff:

1. each branch contained in Φ is maximal
2. any set of branches Ψ such that:
   a. $\Phi \subset \Psi$
   b. $\langle X, A, \Psi\rangle$ is a tableau

   is a redundant superset of Φ.

A tableau is *incomplete* iff it is not complete.

Let us note that, once again, the two variants of the concept are equivalent. This is covered by another fact.

Proposition 3.47. *Let $X \subseteq \mathsf{For}_{\mathbf{TL}}$, $A \in \mathsf{For}_{\mathbf{TL}}$ and Φ be a set of branches. $\langle X, A, \Phi\rangle$ is a complete tableau in variant one 3.36 iff $\langle X, A, \Phi\rangle$ is a complete tableau in variant two 3.46.*

Proof. Let $X \subseteq \mathsf{For}_{\mathbf{TL}}$, $A \in \mathsf{For}_{\mathbf{TL}}$ and Φ be a set of branches.

Assume that $\langle X, A, \Phi\rangle$ is a complete tableau in variant one 3.42. Since by virtue of the definition of tableau 3.36, set Φ contains precisely one branch and according to definition 3.42, that branch is maximal, so each branch contained in set Φ is maximal, which constitutes condition 1 of being a complete tableau in variant two of the definition of complete tableau 3.46. Let us now check if condition 2 of the definition in variant two holds. It claims that for any set of branches Ψ such that:

(a) $\Phi \subset \Psi$

(b) $\langle X, A, \Psi\rangle$ is a tableau

Ψ is a redundant superset of Φ.

So, take any such set of branches Ψ that $\Phi \subset \Psi$. However, by definition of tableau 3.36, triple $\langle X, A, \Psi\rangle$ is not a tableau since Ψ is a set of at least two elements, whereas in a tableau, a set of branches includes precisely one branch. Thus, condition 2 of the definition of complete tableau in variant two 3.46 is satisfied in the empty way.

Now, assume that $\langle X, A, \Phi \rangle$ is a complete tableau in variant two 3.46. According to definition 3.46, each branch contained in Φ is maximal, and what is more, by virtue of definition of tableau 3.36, Φ contains precisely one branch. Thus, a branch contained in Φ is a maximal branch, and consequently, tableau $\langle X, A, \Phi \rangle$ is complete according to the definition of complete tableau in variant one 3.42. □

Evidently, both concepts of a redundant variant of branch and a redundant superset, and their conclusions for the tableau system for **TL** play no role whatsoever in the equivalence proof.

When constructing a complete tableau, we may face a situation where a branch not only becomes maximal, but also finishes with a t-inconsistent set. Such a tableau is called a closed tableau. Let us, again, consider two variants. Variant one to begin with.

Definition 3.48 (Closed/open tableau — variant one)**.** Assume that $\langle X, A, \Phi \rangle$ is a tableau. We shall state that $\langle X, A, \Phi \rangle$ is *closed* iff a branch contained in Φ is closed. A tableau is *open* iff it is not closed.

Variant two, in turn, corresponding to the definition aiming at the general definition of a closed tableau, has the following form.

Definition 3.49 (Closed/open tableau — variant two)**.** Let $\langle X, A, \Phi \rangle$ be a tableau. We shall state that $\langle X, A, \Phi \rangle$ is *closed* iff the below conditions are met:

1. $\langle X, A, \Phi \rangle$ is a complete tableau
2. each branch contained in Φ is closed.

A tableau is *open* iff it is not closed.

Proposition 3.50. *Let $X \subseteq \mathsf{For}_{\mathbf{TL}}$, $A \in \mathsf{For}_{\mathbf{TL}}$ and Φ be a set of branches. $\langle X, A, \Phi \rangle$ is a closed tableau in variant one 3.48 iff $\langle X, A, \Phi \rangle$ is a closed tableau in variant two 3.49.*

Proof. Let $X \subseteq \mathsf{For}_{\mathbf{TL}}$, $A \in \mathsf{For}_{\mathbf{TL}}$ and Φ be a set of branches.

Assume that $\langle X, A, \Phi \rangle$ is a closed tableau in variant one 3.48. Thus, in the light of the definition of complete tableau 3.42, $\langle X, A, \Phi \rangle$ is a complete tableau — since branch which is contained in it is, by conclusion 3.30, maximal — which constitutes condition 1 of being a closed tableau in variant two of the definition of closed tableau 3.49. Since by virtue of the definition of tableau 3.36, set Φ contains precisely one branch and according to definition 3.48, that branch is closed, so each branch contained in set Φ is closed, which constitutes condition 2 of being a closed tableau in variant two of the definition of closed tableau 3.49.

Now, assume that $\langle X, A, \Phi\rangle$ is a closed tableau in variant two 3.49. Thus, by condition 2 of definition 3.49, each branch contained in Φ is closed, and, what is more, by virtue of definition of tableau 3.36, Φ contains precisely one branch, which constitutes a condition of being a closed tableau in variant one of the definition of closed tableau 3.48. □

To sum up, we have shown that specific tableau concepts of the tableau system for **TL** are special cases of more general concepts applied earlier (*modulo* set of tableau rules $\mathbf{R_{TL}}$ and resulting sets of branches), and in the considered cases are equivalent to them.

By virtue of the definition of closed tableau and the definition of complete tableau, and conclusion 3.30, we get another conclusion.

Corollary 3.51. *Each closed tableau is a complete tableau.*

Further, we will show that the concept of tableau is significantly helpful in determining the occurrence of relation $\triangleright$, while this concept, in terms of range, is equal to the concept of implication $\models$.

3.5 Completeness theorem for the tableau system for TL

Let us begin with the definition of model generated by a branch.

Definition 3.52 (Model generated by branch)**.** Let ϕ be any branch. We define the following function $At(\phi) = \bigcup\phi \cap (\mathsf{Te}_{\mathbf{TL}} \setminus \mathsf{For}_{\mathbf{TL}})$.

We shall state that model $\mathfrak{M}_{\mathbf{TL}} = \langle D, d\rangle$ is *generated by branch* ϕ iff:

- $D = \{x \in \mathbb{N} : x \in *(At(\phi))\}$
- for any name latter $P \in \mathsf{Ln}$, $x \in d(P)$ iff $P_{+x} \in At(\phi)$.

From this definition, we get the following conclusion.

Corollary 3.53. *Let ϕ be an open branch. Then, there exists a model generated by ϕ.*

Proof. By definition of open branch 3.28, definition of model generated by branch 3.52 and definition of model 3.3. □

Lemma 3.54 (On generation of model)**.** *Let ϕ be an open and maximal branch. Then, there exists model $\mathfrak{M}_{\mathbf{TL}}$ generated by ϕ such that $\mathfrak{M}_{\mathbf{TL}} \models \bigcup\phi \cap \mathsf{For}_{\mathbf{TL}}$.*

Proof. Take any open and maximal branch ϕ. Since branch ϕ is open, so by previous conclusion 3.53, there exists model $\mathfrak{M}_{\mathbf{TL}} = \langle D, d\rangle$ generated by ϕ.

We will now show that for any formula A contained in $\bigcup\phi$, it is the case that $\mathfrak{M}_{\mathbf{TL}} \models A$, i.e. $\mathfrak{M}_{\mathbf{TL}} \models \bigcup\phi \cap \mathsf{For}_{\mathbf{TL}}$. The proof will be carried out with consideration of all the possible cases of construction of formula A. Now, assume that $A \in \bigcup\phi$. By definition of formula, for some name letters P, $Q \in \mathsf{Ln}$, there must occur one of the following cases.

1. $A = P\mathbf{a}Q$. Take any object $i \in D$ such that $i \in d(P)$. By definition of model generated 3.52, set $\bigcup\phi$ contains tableau expression P_{+i}. Since ϕ is a maximal and open branch, so by virtue of tableau rule $R\mathbf{a}_{+}$, $\bigcup\phi$ also contains tableau expression Q_{+i}. By definition of model generated 3.52, $i \in d(Q)$. Hence, $d(P) \subseteq d(Q)$, and by definition of truth in model 3.4, we thus get that $\mathfrak{M}_{\mathbf{TL}} \models P\mathbf{a}Q$. In turn, if there exists no such $i \in D$ that $i \in d(P)$, then $\varnothing = d(P) \subseteq d(Q)$, so by definition of truth in model 3.4, we get $\mathfrak{M}_{\mathbf{TL}} \models P\mathbf{a}Q$.
2. $A = P\mathbf{i}Q$. Since ϕ is a maximal and open branch, so by virtue of tableau rule $R\mathbf{i}$, set $\bigcup\phi$ also contains tableau expressions P_{+i}, Q_{+i}, for some $i \in \mathbb{N}$. By definition of model generated 3.52, $i \in d(P)$ and $i \in d(Q)$. Since $d(P) \cap d(Q) \neq \varnothing$, so by definition of truth in model 3.4, we get that $\mathfrak{M}_{\mathbf{TL}} \models P\mathbf{i}Q$.
3. $A = P\mathbf{e}Q$. Take any object $i \in D$ such that $i \in d(P)$. By definition of model generated 3.52, set $\bigcup\phi$ contains tableau expression P_{+i}. Since ϕ is a maximal and open branch, so by virtue of tableau rule $R\mathbf{e}_{-}$, $\bigcup\phi$ also contains tableau expression Q_{-i}. Since branch ϕ is open, so expression $Q_{+i} \notin \bigcup\phi$, and consequently, by definition of model generated 3.52, $i \notin d(Q)$. Thus, $d(P) \cap d(Q) = \varnothing$ and by definition of truth in model 3.4, we get $\mathfrak{M}_{\mathbf{TL}} \models P\mathbf{e}Q$. In turn, if there exists no object $i \in D$ such that $i \in d(P)$, then $d(P) \cap d(Q) = \varnothing$, so by definition of truth in model 3.4, we get that $\mathfrak{M}_{\mathbf{TL}} \models P\mathbf{e}Q$.
4. $A = P\mathbf{o}Q$. Since ϕ is a maximal and open branch, so by virtue of tableau rule $R\mathbf{o}$, set $\bigcup\phi$ also contains tableau expressions P_{+i}, Q_{-i}, for some $i \in \mathbb{N}$. By definition of model generated 3.52, $i \in d(P)$ and — since branch ϕ is open and, consequently, expression $Q_{+i} \notin \bigcup\phi$ — $i \notin d(Q)$, so $d(P) \nsubseteq d(Q)$. Thus, by definition of truth in model 3.4, we get $\mathfrak{M}_{\mathbf{TL}} \models P\mathbf{o}Q$. □

We will now move on to a lemma which claims that application of tableau rules for the extension of branches does not reach beyond the model which is appropriate. In other words, according to the definition of model appropriate for the set of expressions 3.16, if we have a set of expressions such that the formulas contained in this set of expressions are true in given model and expressions stating inclusion or non-inclusion of a denotation of given index in the scope of name are interpretable in the model (they do not contradict the state of affairs in the model), then the extension of that set with the use of rules produces a set that still has the above properties.

Lemma 3.55. *Let $\mathfrak{M}_{TL}$ be any model, $X, Y \subseteq \mathsf{Te}_{TL}$, and let $R \in \mathbf{R}_{TL}$. Then, if $\langle X, Y\rangle \in R$ and $\mathfrak{M}_{TL}$ is appropriate for set of expressions X, then $\mathfrak{M}_{TL}$ is appropriate for Y.*

Proof. In the proof, we will make use of definition of model appropriate for the set of expressions 3.16. Let $\mathfrak{M}_{\mathbf{TL}} = \langle D, d\rangle$ be any model and $X, Y \subseteq \mathsf{Te}_{\mathbf{TL}}$. We will consider all cases of rules $R \in \mathbf{R}_{\mathbf{TL}}$, assuming that $\langle X, Y\rangle \in R$ and $\mathfrak{M}_{\mathbf{TL}}$ is appropriate for set of expressions X, and showing that then $\mathfrak{M}_{\mathbf{TL}}$ is appropriate for Y.

1. Let $R = R\mathbf{a}_+$, then $\langle X, Y\rangle = \langle Z \cup \{P\mathbf{a}Q, P_{+i}\}, Z \cup \{P\mathbf{a}Q, P_{+i}, Q_{+i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{TL}}$, $P, Q \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since $\mathfrak{M}_{\mathbf{TL}}$ is appropriate for set of expressions X, so by definition 3.16, $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{a}Q$ and there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that for each name letter $S \in \mathsf{Ln}$ and each $j \in \mathbb{N}$: if $S_{+j} \in X$, then $\gamma(j) \in d(S)$ and if $S_{-j} \in X$, then $\gamma(j) \notin d(S)$; due to the fact that $P_{+i} \in X$, also $\gamma(i) \in d(P)$, while since $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{a}Q$, hence by definition of truth in model 3.4, $\gamma(i) \in d(Q)$, since $d(P) \subseteq d(Q)$; consequently, by definition of model appropriate for the set of expressions 3.16, model $\mathfrak{M}_{\mathbf{TL}}$ is appropriate for set of expressions $Y = Z \cup \{P\mathbf{a}Q, P_{+i}, Q_{+i}\}$.
2. Let $R = R\mathbf{i}$, then $\langle X, Y\rangle = \langle Z \cup \{P\mathbf{i}Q\}, Z \cup \{P\mathbf{i}Q, P_{+i}, Q_{+i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{TL}}$, $P, Q \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since $\mathfrak{M}_{\mathbf{TL}}$ is appropriate for set of expressions X, so by definition 3.16, $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{i}Q$ and there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that for each name letter $S \in \mathsf{Ln}$ and each $j \in \mathbb{N}$: if $S_{+j} \in X$, then $\gamma(j) \in d(S)$ and if $S_{-j} \in X$, then $\gamma(j) \notin d(S)$; however, rule $R\mathbf{i}$ enriches set X with expressions P_{+i}, Q_{+i} and index i is new, it has not occurred in any expression from set X, while since $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{i}Q$, so by virtue of definition of truth in model 3.4, in the domain there exists certain object x such that $x \in d(P) \cap d(Q)$; so we define function $\gamma' : \mathbb{N} \longrightarrow D$ such that for any $k \in \mathbb{N}$, if $k \neq i$, then $\gamma'(k) = \gamma(k)$ and $\gamma'(i) = x$, consequently, by definition of model appropriate for the set of expressions 3.16, model $\mathfrak{M}_{\mathbf{TL}}$ is appropriate for set of expressions $Y = Z \cup \{P\mathbf{i}Q, P_{+i}, Q_{+i}\}$.
3. Let $R = R\mathbf{e}_-$, then $\langle X, Y\rangle = \langle Z \cup \{P\mathbf{e}Q, P_{+i}\}, Z \cup \{P\mathbf{e}Q, P_{+i}, Q_{-i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{TL}}$, $P, Q \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since $\mathfrak{M}_{\mathbf{TL}}$ is appropriate for set of expressions X, so by definition 3.16, $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{e}Q$ and there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that for each name letter $S \in \mathsf{Ln}$ and each $j \in \mathbb{N}$: if $S_{+j} \in X$, then $\gamma(j) \in d(S)$ and if $S_{-j} \in X$, then $\gamma(j) \notin d(S)$; due to the fact that $P_{+i} \in X$, also $\gamma(i) \in d(P)$, while since $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{e}Q$, hence by virtue of definition of truth in model 3.4 $\gamma(i) \notin d(Q)$, since $d(P) \cap d(Q) = \varnothing$; consequently, by definition of model appropriate for the set of expressions 3.16, model $\mathfrak{M}_{\mathbf{TL}}$ is appropriate for set of expressions $Y = Z \cup \{P\mathbf{e}Q, P_{+i}, Q_{-i}\}$.

4. Let $R = R\mathbf{o}$, then $\langle X, Y\rangle = \langle Z \cup \{P\mathbf{o}Q\}, Z \cup \{P\mathbf{o}Q, P_{+i}, Q_{-i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{TL}}$, P, $Q \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since $\mathfrak{M}_{\mathbf{TL}}$ is appropriate for set of expressions X, so by definition 3.16, $\mathfrak{M}_{\mathbf{TL}} \models P\mathbf{o}Q$ and there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that for each name letter $S \in \mathsf{Ln}$ and each $j \in \mathbb{N}$: if $S_{+j} \in X$, then $\gamma(j) \in d(S)$ and if $S_{-j} \in X$, then $\gamma(j) \notin d(S)$; however, rule $R\mathbf{o}$ enriches set X with expressions P_{+i}, Q_{-i} and index i is new, it has not occurred in any expression from set X, while since $\mathfrak{M}_{\mathbf{TL}} \models P\mathbf{o}Q$, so by virtue of definition of truth in model 3.4, in the domain there exists certain object x such that $x \in d(P)$, but $x \notin d(Q)$; so we define function $\gamma' : \mathbb{N} \longrightarrow D$ such that for any $k \in \mathbb{N}$, if $k \neq i$, then $\gamma'(k) = \gamma(k)$ and $\gamma'(i) = x$, consequently, by definition of model appropriate for the set of expressions 3.16, model $\mathfrak{M}_{\mathbf{TL}}$ is appropriate for set of expressions $Y = Z \cup \{P\mathbf{o}Q, P_{+i}, Q_{-i}\}$. □

We will now proceed to the main theorem which synthesizes all so far covered facts and lemmas, stating the dependencies between the semantic consequence relation, branch consequence relation and the existence of closed tableau.

Theorem 3.56 (Theorem on the completeness of tableau system for **TL**). *For any $X \subseteq \mathsf{For}_{\mathbf{TL}}$, $A \in \mathsf{For}_{\mathbf{TL}}$, the below statements are equivalent.*

- $X \models A$
- $X \rhd A$
- *there exists finite $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi\rangle$.*

Proof. Take any $X \subseteq \mathsf{For}_{\mathbf{TL}}$ and $A \in \mathsf{For}$. We will prove three implications.

(a) $X \models A \Rightarrow X \rhd A$

Assume that $X \not\rhd A$. We must show that $X \not\models A$. From the assumption and definition $\rhd$ 3.33, we know that for each finite set $Y \subseteq X$, there exists a branch beginning with set $Y \cup \{\circ(A)\}$ which is maximal and open. Take any finite subset $Y' \subseteq X$. Thus, there exists branch ϕ beginning with set $Y' \cup \{\circ(A)\}$ which is maximal and open. Since branch ϕ is maximal and open, so by lemma on generation of model 3.54, there exists model $\mathfrak{M}_{\mathbf{TL}}$ such that $\mathfrak{M}_{\mathbf{TL}} \models Y' \cup \{\circ(A)\}$.

Due to the fact that set Y' is arbitrary, so for any Y, finite subset of X, there exists model $\mathfrak{M}_{\mathbf{TL}}$ such that $\mathfrak{M}_{\mathbf{TL}} \models Y \cup \{\circ(A)\}$. Thus, for any Y, finite subset of X, $Y \not\models A$, due to the fact 3.13. While by fact 3.9, relation $\models$ is compact, so $X \not\models A$.

(b) $X \rhd A \Rightarrow$ there exists finite $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi\rangle$

Assume that for any finite subset $Y \subseteq X$, each tableau $\langle Y, A, \Phi\rangle$ is open. Take any finite subset $Y \subseteq X$. By the assumption, each tableau $\langle Y, A, \Phi\rangle$ is open. By definition of tableau 3.36, each set Φ contains one branch beginning with set

$Y \cup \{\circ(A)\}$. Consequently, by definition of open tableau 3.48, each branch beginning with set $Y \cup \circ(A)$ is open. However, from fact 3.32, we know that since set $Y \cup \{\circ(A)\}$ is finite, then there exists branch beginning with $Y \cup \{\circ(A)\}$ that is maximal. Since that branch is open and set Y is a finite subset of X, by definition of relation of branch consequence 3.33 $X \not\triangleright A$,

(c) there exists finite $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi \rangle \Rightarrow X \vDash A$

Assume that there exists finite subset $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi \rangle$. Indirectly assume that $X \nvDash A$, thus, by definition of relation of semantic consequence 3.5, there exists such model $\mathfrak{M}_{\mathbf{TL}}$ that $\mathfrak{M}_{\mathbf{TL}} \vDash X$, but $\mathfrak{M}_{\mathbf{TL}} \nvDash A$. From fact 3.13, we know that $\mathfrak{M}_{\mathbf{TL}} \vDash \circ(A)$. Consequently, $\mathfrak{M}_{\mathbf{TL}} \vDash X \cup \{\circ(A)\}$, and thus $\mathfrak{M}_{\mathbf{TL}} \vDash Y \cup \{\circ(A)\}$. Since tableau $\langle Y, A, \Phi \rangle$ is closed, then by definition 3.48, Φ contains branch ψ beginning with set $Y \cup \{\circ(A)\}$ which is closed. So, branch ψ is maximal by conclusion 3.30 and has lengths n, for certain $n \in \mathbb{N}$. What is more, by conclusion 3.29, set $\psi(n)$ is t-inconsistent.

Since $\mathfrak{M}_{\mathbf{TL}} \vDash Y \cup \{\circ(A)\}$, so model $\mathfrak{M}_{\mathbf{TL}}$ is appropriate for set $Y \cup \{\circ(A)\}$. Now, applying lemma 3.55, $n-1$-times we get conclusion that $\mathfrak{M}_{\mathbf{TL}}$ is appropriate for set $\psi(n)$. However, due to the fact that $\psi(n)$ is t-inconsistent and conclusion 3.17, there exists no model appropriate for $\psi(n)$. □

3.5.1 Estimation of cardinality of model for TL

When applying tableau methods for **TL** another issue appears. With the tableau proof, we can estimate the upper limit of the cardinality of models that we only need to check in order to establish whether a given inference is correct[7]. While, in the study we do not take up this issue in general (as it is related to the issue of decidaility), but this outcome for **TL** we can virtually get directly from the theorem on completeness of tableau system for **TL** 3.56.[8]

By *existential formula*, we mean any formula in form P**i**Q or P**o**Q, where P, Q, $\in$ Ln.

Now, we shall define function $\lambda : P(\mathsf{For}_{\mathbf{TL}}) \longrightarrow P(\mathsf{For}_{\mathbf{TL}})$ with the following condition: for any set $\Phi \in P(\mathsf{For}_{\mathbf{TL}})$, $\lambda(\Phi) = \{x \in \Phi : x \text{ is an existential formula}\}$. So, from each set of formulas, function λ "selects" all existential formulas that belong to a given set.

---

7 Estimations of cardinality of model for syllogistic, for languages richer than the language of **TL**, have been examined in the studies by: A. Pietruszczak [20], [21], P. Kulicki [17], [18].

8 This outcome was originally described in article [9]. However, when defining the tableau system for **TL**, in that study, we applied another set of rules and non-formalised tableau concepts.

Now, in turn, we shall define function $\sigma : \{\Psi \in P(\mathsf{For}_{\mathbf{TL}}) : \Psi \text{ is a finite set}\} \longrightarrow \mathbb{N}$ with the following condition: for any finite set $\Psi \in P(\mathsf{For}_{\mathbf{TL}})$, $\sigma(\Psi) = |\lambda(\Psi)|$. So, function σ "counts" the number of existential formulas that are found in any finite set of formulas.

We have the theorem.

Theorem 3.57. *Let X be a finite set of formulas and let $A \in \mathsf{For}_{\mathbf{TL}}$. Then:*
$\forall_{\mathfrak{M}_{TL}=\langle D,d\rangle} (|D| = \sigma(X \cup \{\circ(A)\}) \Rightarrow (\mathfrak{M}_{TL} \vDash X \Rightarrow \mathfrak{M}_{TL} \vDash A))$ *iff* $X \vDash A$.

Proof. Take any finite set of formulas X and any formula A.

The implication "from the right to the left" follows from the definition of relation of semantic consequence 3.5. Because if $X \vDash A$, then for any model $\mathfrak{M}_{\mathbf{TL}}$, if $\mathfrak{M}_{\mathbf{TL}} \vDash X$, then $\mathfrak{M}_{\mathbf{TL}} \vDash A$. Particularly, for such models $\mathfrak{M}_{\mathbf{TL}} = \langle D, d\rangle$ that $|D| = \sigma(X \cup \{\circ(A)\})$.

Now, assume that $X \nvDash A$. From the theorem on completeness of tableau system of **TL** 3.56, we get that $X \not\triangleright A$. By definition of relation of branch consequence 3.33, for any finite $Y \subseteq X$, there exists maximal and open branch beginning with set $Y \cup \{\circ(A)\}$. Since X is a finite set, so there exists maximal and open branch ϕ beginning with set $X \cup \{\circ(A)\}$. So, by lemma on generation of model 3.54, there exists such model $\mathfrak{M}_{\mathbf{TL}} = \langle D, d\rangle$ that $\mathfrak{M}_{\mathbf{TL}} \vDash X \cup \{\circ(A)\}$, thus $\mathfrak{M}_{\mathbf{TL}} \vDash X$ and $\mathfrak{M}_{\mathbf{TL}} \nvDash A$, by fact 3.13.

By definition 3.52, domain $D = \{i \in \mathbb{N} : i \in *(At(\phi))\}$, and each object $i \in D$ emerged in some expression in $\bigcup\phi \cap (\mathsf{Te}_{\mathbf{TL}} \setminus \mathsf{For})$ by the application of rule *R***i** or *R***o** to some existential formula. As for each existential formula, we can only one time apply rule *R***i** or *R***o**, thus $|D| \leq \sigma(X \cup \{\circ(A)\})$.

Model $\mathfrak{M}_{\mathbf{TL}}$ was generated by an open and maximal branch, so rule *R***i** or *R***o** was applied to each existential formula. Thus, for any existential formula, there exists at least one object $i \in D$, due to the fact that rules *R***i** and *R***o** introduce expressions with new indices. Hence, $|D| \geq \sigma(X \cup \{\circ(A)\})$.

So, consequently, there exists such model $\mathfrak{M}_{\mathbf{TL}} = \langle D, d\rangle$ that $|D| = \sigma(X \cup \{\circ(A)\})$, $\mathfrak{M}_{\mathbf{TL}} \vDash X$ and $\mathfrak{M}_{\mathbf{TL}} \nvDash A$. □