4 Tableau system for modal logic S5

4.1 Introductory remarks

Chapter Four will be devoted to a case extremely different than, presented in Chapter One, the case of tableau system for **CPL**. For we consider modal logic **S5** (for short: **S5**), defining the set of tableau expressions in such a way for it neither to concur with the set of formulas of logic **S5**, nor to be its proper superset — both sets are disjoint[1]. The proofs in the tableau system that we are going to construct will therefore be carried out in a language different from the one in which we want to determine whether or not a given consequence relation holds.

Another important difference between the previous systems and the one currently being defined is that the systems described previously featured the property of a finite branch, which is not the case for the presented tableau system for **S5**.

Therefore, it may happen that when constructing a branch and consequently a tableau which begins with a finite set of expressions, it is not possible to finish it as some sequences of application of the rules become cyclical. The lack of a finite branch property forces changes in a concept of maximal branch and in dependent concepts.

So, the tableau concepts defined in previous chapters will become special cases of tableau concepts for systems that do not feature the property of a finite branch. The leading change is the generalisation of the concept of maximal branch. Still, the maximal branch is a branch to which no rule can be applied anymore in order to extend it as it contains everything tableau rules are capable of introducing to it. Previously, however, for the cases of finite sets this meant that the maximal branch was of a finite length. It does not have to be the case this time. A maximal branch can be infinite even though the infinity of a branch does not imply its maximal nature. For there exist cases of infinite-length branches that are not maximal.

It is worth noting that the systems that feature the property of a finite branch are decidable. For theoretically, always in a finite number of steps it is possible to construct a complete tableau for them — closed or open one, thus answering the question whether a given formula is or is not tableau derivable on the grounds of given premises. In the case of tableau systems that do not feature a finite branch

1 Some of the findings we will present here were described in English-language article [8]. In particular, we took up the general definition of concepts for modal logics defined by the semantics of possible worlds, to which we will return in the final chapter of the book.

property, these systems may not be decidable. So, although we prove that they are complete and consistent in relation to the initial, semantically defined relation, there does not have to exist a general way of constructing a complete tableau even though such a tableau can exist.

So, we treat the case of a tableau system for **S5** logic as a model for two reasons:

- the set of tableau expressions is naturally different than the set of formulas
- branches beginning with finite sets of expressions can feature an infinite lengths.

Both reasons lead to the need of generalization of the so-far applied tableau concepts, affecting the formalisation of tableau methods offered in the book.

4.2 Language and semantics

As usual, for the start we will define the basic concepts for the logic **S5**. First, we will take up the language of **S5**.

Definition 4.1 (Alphabet of **S5**). *Alphabet of the modal logic* **S5** is the sum of the following sets:

- set of logical constants: $\mathsf{Lc} = \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \Diamond, \Box\}$
- set of propositional letters: $\mathsf{Var} = \{p_1, q_1, r_1, p_2, q_2, r_2, \ldots\}$
- set of brackets: $\{), (\}$.

Although the set of propositional letters is infinite and includes indexed letters, in practice we will use a finite number of the following letters: p, q, r, s.

Definition 4.2 (Formula **S5**). *Set of formulas of modal logic* **S5** is the smallest set X which meets conditions:

1. $\mathsf{Var} \subseteq X$
2. if $A, B \in X$, then
   a. $\neg A \in X$
   b. $\Box A \in X$
   c. $\Diamond A \in X$
   d. $(A \wedge B) \in X$
   e. $(A \vee B) \in X$
   f. $(A \rightarrow B) \in X$
   g. $(A \leftrightarrow B) \in X$.

We specify this set as $\mathsf{For}_{\mathbf{S5}}$, and its elements will be called *formulas*.

We will now proceed to the interpretation of set $\mathsf{For}_{\mathbf{S5}}$. To begin with, let us recall some properties of binary relations.

Definition 4.3. Let R be a binary relation defined on Cartesian product $X \times X$, for some set X. We shall state that:

1. R is a *universal* relation iff $\forall_{x,y \in X}\, xRy$
2. R is an *equivalence* relation iff
   a. R is a reflexive relation, i.e. $\forall_{x \in X}\, xRx$
   b. R is a symmetric relation, i.e. $\forall_{x,y \in X}\, (xRy \Rightarrow yRx)$
   c. R is a transitive relation, i.e. $\forall_{x,y,z \in X}\, (xRy \,\&\, yRz \Rightarrow xRz)$.

From the above definition 4.3, the following conclusion follows.

Corollary 4.4. *Each universal relation is an equivalence relation.*

We will now proceed to the concept of model for formulas **S5**.

Definition 4.5 (Model for language of **S5**). *Model* $\mathfrak{M}_{\mathbf{S5}}$ *for language of* ***S5*** will be called such ordered quadruple $\langle W, R, V, w \rangle$ that:

- W is a non-empty set
- R is a universal relation defined on Cartesian product $W \times W$, i.e. $R = W \times W$
- V is a function valuating propositional letters in the elements of set W, i.e. $V : \mathsf{Var} \times W \longrightarrow \{0,1\}$
- $w \in W$.

From definition of model 4.5 and conclusion 4.4 another conclusion results.

Corollary 4.6. *Each model with a universal relation is a model with an equivalence relation.*

Models with equivalence relation will be denoted as $\mathfrak{M}'_{\mathbf{S5}}$.

Now, we shall define truth in model. This definition also applies to models with equivalence relation $\mathfrak{M}'_{\mathbf{S5}}$.

Definition 4.7 (Truth in model). Let $\mathfrak{M}_{\mathbf{S5}} = \langle W, R, V, w \rangle$ be a model and let $A \in \mathsf{For}_{\mathbf{S5}}$. Formula A is *true* in model $\mathfrak{M}_{\mathbf{S5}}$ (for short: $\mathfrak{M}_{\mathbf{S5}} \models A$) iff for any formulas $B, C \in \mathsf{For}_{\mathbf{S5}}$ the below conditions are met:

1. if $A \in \mathsf{Var}$, then $V(A, w) = 1$
2. if $A := \neg B$, then formula B is not true in model $\mathfrak{M}_{\mathbf{S5}}$ (for short: $\mathfrak{M}_{\mathbf{S5}} \not\models B$)
3. if $A := (B \wedge C)$, then $\mathfrak{M}_{\mathbf{S5}} \models B$ and $\mathfrak{M}_{\mathbf{S5}} \models C$
4. if $A := (B \vee C)$, then $\mathfrak{M}_{\mathbf{S5}} \models B$ or $\mathfrak{M}_{\mathbf{S5}} \models C$
5. if $A := (B \rightarrow C)$, then $\mathfrak{M}_{\mathbf{S5}} \not\models B$ or $\mathfrak{M}_{\mathbf{S5}} \models C$
6. if $A := (B \leftrightarrow C)$, then $\mathfrak{M}_{\mathbf{S5}} \models B$ iff $\mathfrak{M}_{\mathbf{S5}} \models C$

7. if $A := \Box B$, then $\forall_{u \in W}(wRu \Rightarrow \langle W, R, V, u\rangle \vDash B)$
8. if $A := \Diamond B$, then $\exists_{u \in W}(wRu \,\&\, \langle W, R, V, u\rangle \vDash B)$.

Formula A is *false* in model $\mathfrak{M}_{\mathbf{S5}}$ (for short: $\mathfrak{M}_{\mathbf{S5}} \nvDash A$) iff it not true.

Let $X \subseteq \mathsf{For}_{\mathbf{S5}}$. Set of formulas X is *true* in model $\mathfrak{M}_{\mathbf{S5}}$ (for short: $\mathfrak{M}_{\mathbf{S5}} \vDash X$) iff for any formula $A \in X$, $\mathfrak{M}_{\mathbf{S5}} \vDash A$. Set of formulas X is *false* in model $\mathfrak{M}_{\mathbf{S5}}$ (for short: $\mathfrak{M}_{\mathbf{S5}} \nvDash X$) iff it is not the case that $\mathfrak{M}_{\mathbf{S5}} \vDash X$.

Now, we will show a fact that displays an connection between models with equivalence relation and those with universal relation.

Proposition 4.8. *For any model* $\mathfrak{M}'_{\mathbf{S5}} = \langle W', R', V', w'\rangle$, *there exists such model* $\mathfrak{M}_{\mathbf{S5}} = \langle W, R, V, w\rangle$ *that for any formula* $A \in \mathsf{For}_{\mathbf{S5}}$, $\mathfrak{M}'_{\mathbf{S5}} \vDash A$ *iff* $\mathfrak{M}_{\mathbf{S5}} \vDash A$.

Proof. Take any model with equivalence relation $\mathfrak{M}'_{\mathbf{S5}} = \langle W', R', V', w'\rangle$. Next, define model with universal relation $\mathfrak{M}_{\mathbf{S5}} = \langle W, R, V, w\rangle$ as follows:

- $W = \{x \in W' : w'Rx\}$
- $R = \{\langle x, y\rangle : x, y \in W\}$
- $V = V'_W$, where V'_W is a restriction of function V' to set $\mathsf{Var} \times W$
- $w = w'$.

R' is an equivalence relation, $R \subseteq R'$, $R = R'_W$, where R'_W constitutes a restriction of relation R' to set $W = \{x \in W' : w'Rx\}$, so R'_W is a universal relation. Since relation R' is reflexive, so obviously $w' \in W$, thus model $\mathfrak{M}_{\mathbf{S5}}$ is well defined.

Consider any propositional letter $q \in \mathsf{Var}$. By definition of model $\mathfrak{M}_{\mathbf{S5}}$, since $V = V'_W$, so for any $u \in W$ it is the case that $(*)$ $V'(q, u) = 1$ iff $V(q, u) = 1$.

Now, take any formula $A \in \mathsf{For}_{\mathbf{S5}}$. We will consider various construction possibilities for formula A, carrying out an inductive proof in respect of the complexity of formula A and showing that for any $u \in W$, the below thesis occurs:

$(**)$ $\langle W', R', V', u\rangle \vDash A$ iff $\langle W, R, V, u\rangle \vDash A$.

Initial step. Take any $u \in W$. Assume that $A \in \mathsf{Var}$. By $(*)$ and definition of truth in model 4.7, we get $\langle W', R', V', u\rangle \vDash A$ iff $\langle W, R, V, u\rangle \vDash A$.

Induction step. Take any formulas $B, C \in \mathsf{For}_{\mathbf{S5}}$ and assume that $(***)$ for B, C and for any $u \in W$, the following occurs:

$\langle W', R', V', u\rangle \vDash B$ iff $\langle W, R, V, u\rangle \vDash B$

$\langle W', R', V', u\rangle \vDash C$ iff $\langle W, R, V, u\rangle \vDash C$.

Take any $u \in W$ and consider the following cases:

1. $A = \neg B$, $A = (B \wedge C)$, $A = (B \vee C)$, $A = (B \rightarrow C)$ or $A = (B \leftrightarrow C)$; then by virtue of assumption $(***)$ and definition of truth in model 4.7, we get $\langle W', R', V', u\rangle \vDash A$ iff $\langle W, R, V, u\rangle \vDash A$

2. $A = \Box B$; then, by virtue of definition of truth in model 4.7, $\langle W', R', V', u\rangle \vDash \Box B$ iff $\forall_{z\in W'}(uR'z \Rightarrow \langle W', R', V', z\rangle \vDash B)$, by definition of set W and relation R and by assumption $(***)$, it is the case iff $\forall_{z\in W}(uRz \Rightarrow \langle W, R, V, z\rangle \vDash B)$, while by virtue of definition of truth in model 4.7, iff $\langle W, R, V, u\rangle \vDash \Box B$, thus $\langle W', R', V', u\rangle \vDash A$ iff $\langle W, R, V, u\rangle \vDash A$
3. $A = \Diamond B$; then, by virtue of definition of truth in model 4.7, $\langle W', R', V', u\rangle \vDash \Diamond B$ iff $\exists_{z\in W'}(uR'z \& \langle W', R', V', z\rangle \vDash B)$, by definition of set W and relation R and by assumption $(***)$, it is the case iff $\exists_{z\in W}(uRz \& \langle W, R, V, z\rangle \vDash B)$, while by virtue of definition of truth in model 4.7, iff $\langle W, R, V, u\rangle \vDash \Diamond B$, thus $\langle W', R', V', u\rangle \vDash A$ iff $\langle W, R, V, u\rangle \vDash A$.

Consequently, we get thesis:

for any $u \in W$, $\langle W', R', V', u\rangle \vDash A$ iff $\langle W, R, V, u\rangle \vDash A$. However, since $w = w'$ and $w \in W$, $\mathfrak{M}'_{\mathbf{S5}} \vDash A$ iff $\mathfrak{M}_{\mathbf{S5}} \vDash A$. □

Making use of the concept of model, we can now define the concept of semantic consequence relation in **S5**. For the entire class of models with universal relation, in a normal way on set $P(\mathsf{For}_{\mathbf{S5}}) \times \mathsf{For}_{\mathbf{S5}}$ we define the consequence relation.

Definition 4.9 (Semantic consequence of **S5**). Let $A \in \mathsf{For}_{\mathbf{S5}}$ and $X \subseteq \mathsf{For}_{\mathbf{S5}}$. We shall state that from set X *follows* formula A (for short: $X \vDash A$) iff for any model $\mathfrak{M}_{\mathbf{S5}}$, if $\mathfrak{M}_{\mathbf{S5}} \vDash X$, then $\mathfrak{M}_{\mathbf{S5}} \vDash A$. Relation $\vDash$ will also be called *semantic consequence relation of logic* **S5** or for short *semantic consequence relation*.

Denotation 4.10. For any set of formulas X and any formula A notation $X \nvDash A$ will mean that it is not the case that $X \vDash A$.

Let us remind that we defined models for **S5** with the use of universal relations. It is known, however, we could define models for **S5** using equivalence relations — since both classes of models determine the same semantic relation of consequence, which is expressed by another fact.

Proposition 4.11. *Let $A \in \mathsf{For}_{S5}$ and $X \subseteq \mathsf{For}_{S5}$. Then, $X \vDash A$ iff for any model with equivalence relation $\mathfrak{M}'$, if $\mathfrak{M}' \vDash X$, then $\mathfrak{M}' \vDash A$.*

Proof. Let $A \in \mathsf{For}_{\mathbf{S5}}$ and $X \subseteq \mathsf{For}_{\mathbf{S5}}$.

First, we will prove the implication "from the left to the right". Assume that $X \vDash A$ and take any model with equivalence relation $\mathfrak{M}'_{\mathbf{S5}} = \langle W, R, V, w\rangle$ such that $\mathfrak{M}'_{\mathbf{S5}} \vDash X$. From fact 4.8 we know that for certain model with universal relation $\mathfrak{M}_{\mathbf{S5}}$ it is the case that for any formula $B \in \mathsf{For}_{\mathbf{S5}}$, $\mathfrak{M}'_{\mathbf{S5}} \vDash B$ iff $\mathfrak{M}_{\mathbf{S5}} \vDash B$. Thus $\mathfrak{M}_{\mathbf{S5}} \vDash X$, and consequently, by assumption $\mathfrak{M}_{\mathbf{S5}} \vDash A$. Again, making use of fact 4.8, we get $\mathfrak{M}'_{\mathbf{S5}} \vDash A$.

We prove implication 'from the left to the right' with the use of conclusion 4.6. Assume that for any model with equivalence relation $\mathfrak{M}'$, if $\mathfrak{M}' \vDash X$, then $\mathfrak{M}' \vDash A$, and take any model $\mathfrak{M}_{\mathbf{S5}}$ such that $\mathfrak{M}_{\mathbf{S5}} \vDash X$. But, by conclusion 4.6, model $\mathfrak{M}_{\mathbf{S5}}$ is a model of equivalence relation, thus $\mathfrak{M}_{\mathbf{S5}} \vDash A$. □

The above issue will be discussed in subchapter devoted to the axiomatization with tableau rules as it directly suggests the possibility of two different axiomatizations.

Let us now go to the concept of contradictory set of formulas.

Definition 4.12. Let $X \subseteq \mathsf{For}_{\mathbf{S5}}$. Set of formulas X is *contradictory* iff for any model $\mathfrak{M}_{\mathbf{S5}} \nvDash X$. Set X is *non-contradictory* iff X it is not contradictory.

Another conclusion follows from definition of contradictory set of formulas 4.12 and definition of truth in model 4.7.

Proposition 4.13. *Let $X \subseteq \mathsf{For}_{\mathbf{S5}}$ and $A \in \mathsf{For}_{\mathbf{S5}}$. If $\{A, \neg A\} \subseteq X$, then X is contradictory.*

4.3 Basic concepts of the tableau system for S5

Unlike the tableau system for **CPL**, in the case of tableau system for **TL** the tableau proofs were carried out in a more rich language than the set of formulas. In the case of tableau system for **S5**, the language of tableau proof has no common elements with the language of logic **S5** — thus in this respect, the described system constitutes a completely new case.

Let us now define the proof language for the tableau system for logic **S5**.

Definition 4.14 (Tableau expressions for **S5**). *Set of tableau expressions* is the union of two following sets:

- Cartesian product: $\mathsf{For}_{\mathbf{S5}} \times \mathbb{N}$
- $\{irj : i,j \in \mathbb{N}\}$.

We specify this set as $\mathsf{Te}_{\mathbf{S5}}$, and its elements will be called *tableau expressions* or simply *expressions*. Numbers present in the expressions will be called *indices*.

In the event of tableau systems for the modal logic, sometimes there is a need of "selecting" indices from the tableau expressions. Thus, let us define an appropriate function to enable the above. Before that, however, we will introduce an auxiliary function $h : \mathsf{Te}_{\mathbf{S5}} \longrightarrow P(\mathbb{N})$ defined for any $A \in \mathsf{For}_{\mathbf{S5}}$ and $i,j \in \mathbb{N}$ with conditions:

- $h(\langle A,i \rangle) = \{i\}$
- $h(irj) = \{i,j\}$.

Definition 4.15 (Function selecting indices)**.** *Function selecting indices* will be called function $* : P(\mathsf{Te}_{\mathbf{S5}}) \longrightarrow P(\mathbb{N})$ defined for any $X \subseteq \mathsf{Te}_{\mathbf{S5}}$ with condition:

- $*(X) = \bigcup\{h(y) : y \in X\}$.

For any subset $\mathsf{Te}_{\mathbf{S5}}$, function $*$ selects all indices present in the expressions from that set.

Let us now proceed to the concept of similar sets of expressions. Intuitively, two sets of expressions are similar iff their expressions contain exactly the same formulas and all expressions in both sets are structurally similar with respect to the indices. Formally:

Definition 4.16 (Similar set of expressions)**.** Let X, $Y \subseteq \mathsf{Te}_{\mathbf{S5}}$. X is *similar* to Y iff there exists such bijection $g : *(X) \longrightarrow *(Y)$ that for any $A \in \mathsf{For}_{\mathbf{S5}}$ and indices $i, j \in \mathbb{N}$:

- $\langle A, i\rangle \in X$ iff $\langle A, g(i)\rangle \in Y$
- $irj \in X$ iff $g(i)rg(j) \in Y$.

Based on definition 4.16, we can draw the following conclusion.

Corollary 4.17. *The relation of similarity is symmetric, i.e. For any sets of expressions X, Y, if set X is similar to Y, then Y is similar to set X.*

Proof. By definition of similar sets of expressions 4.16, by the fact that function g is bijection and by the equivalences present in both conditions. □

Next, we introduce a definition of tableau inconsistent set.

Definition 4.18 (Tableau inconsistent set of expressions)**.** Set $X \subseteq \mathsf{Te}_{\mathbf{S5}}$ will be called *tableau inconsistent* (for short: t-inconsistent) iff for some formula $A \in \mathsf{For}_{\mathbf{S5}}$ and some index $i \in \mathbb{N}$, $\langle A, i\rangle \in X$ and $\langle \neg A, i\rangle \in X$. Set X is *tableau consistent* (for short: t-consistent) iff X is not t-inconsistent.

From this definition, the following conclusion results.

Corollary 4.19. *Let X, $Y \subseteq \mathsf{Te}_{\mathbf{S5}}$. If set X is similar to set Y, then X is t-consistent iff Y is t-consistent.*

Proof. Let X, $Y \subseteq \mathsf{Te}_{\mathbf{S5}}$, and assume that X is similar to set Y. Also, assume that set X is t-inconsistent. Then, by definition of t-inconsistent set 4.18, set X contains expressions $\langle A, i\rangle$, $\langle \neg A, i\rangle$, for some $A \in \mathsf{For}_{\mathbf{S5}}$ and $i \in \mathbb{N}$. By definition of similar set 4.16, there exists bijection $g : *(X) \longrightarrow *(Y)$ and set Y contains expressions $\langle A, g(i)\rangle$, $\langle \neg A, g(i)\rangle$, thus by definition of t-inconsistent set 4.18, set Y

is t-inconsistent. On the other hand, since relation of set similarity is symmetric 4.17, then by assumption, set Y is similar to set X. Also, assume that set Y is t-inconsistent. Then, by definition of t-inconsistent set 4.18, set Y contains expressions $\langle A, i\rangle$, $\langle \neg A, i\rangle$, for some $A \in \mathsf{For}_{\mathbf{S5}}$ and $i \in \mathbb{N}$. By definition of similar set 4.16, there exists bijection $g' : *(Y) \longrightarrow *(X)$ and set X contains expressions $\langle A, g'(i)\rangle$, $\langle \neg A, g'(i)\rangle$, thus by definition of t-inconsistent set 4.18, set X is t-inconsistent. □

For further studies, we also need a concept that would combine models with the set of expressions.

Definition 4.20 (Model appropriate for set of expressions)**.** Let $\mathfrak{M}_{\mathbf{S5}} = \langle W, R, V, w\rangle$ be a model and let $X \subseteq \mathsf{Te}_{\mathbf{S5}}$. We shall state that model $\mathfrak{M}_{\mathbf{S5}}$ is *appropriate* for X iff there exists such function $f : \mathbb{N} \longrightarrow W$, that for any $A \in \mathsf{For}_{\mathbf{S5}}$ and $i, j \in \mathbb{N}$:

- if $\langle A, i\rangle \in X$, then $\langle W, R, V, f(i)\rangle \vDash A$
- if $irj \in X$, then $f(i)Rf(j)$.

Another fact follows from the above definition.

Proposition 4.21. *Let X be a t-inconsistent set of expressions. Then, there exists no model $\mathfrak{M}_{\mathbf{S5}}$ appropriate for X.*

Proof. Take any set of expressions X and any model $\mathfrak{M}_{\mathbf{S5}} = \langle W, R, V, w\rangle$ and assume that X is t-inconsistent. Then, by definition of tableau inconsistent set of expressions 4.18, for some formula $A \in \mathsf{For}_{\mathbf{S5}}$ and for some index $i \in \mathbb{N}$, $\langle A, i\rangle \in X$ and $\langle \neg A, i\rangle \in X$. If model $\mathfrak{M}_{\mathbf{S5}}$ were appropriate for set of expressions X, then by definition of model appropriate for the set of expressions 4.20, there would exist such function $f : \mathbb{N} \longrightarrow W$ that if $\langle A, i\rangle \in X$, then $\langle W, R, V, f(i)\rangle \vDash A$ and if $\langle \neg A, i\rangle \in X$, then $\langle W, R, V, f(i)\rangle \vDash \neg A$, and so $\langle W, R, V, f(i)\rangle \vDash A$ and $\langle W, R, V, f(i)\rangle \vDash \neg A$. However, from fact 4.13 and definition 4.12, it follows that there exists no such a model. Hence, model $\mathfrak{M}_{\mathbf{S5}}$ is not appropriate for set of expressions X. □

4.3.1 Tableau rules for S5

As in the previous cases of described tableau systems, we first provide the general concept of rule. Not only because it facilitates provision of the general features that a tableau rule must meet. In the case of system construction for **S5** we will provide — as in the case of **TL** — alternative sets of tableau rules that are suitable for construction of tableau system for logic **S5**. This means that within the below general concept of rule, we can define different sets of tableau rules that define various, however equivalent in terms of scope of branch consequence, tableau systems for **S5** (see note 2.19 and 3.20).

Definition 4.22 (Rule). Let $P(\mathsf{Te_{S5}})$ be a power set of the set of tableau expressions. Let $P(\mathsf{Te_{S5}})^n$ be n-ary Cartesian product $\underbrace{P(\mathsf{Te_{S5}}) \times \cdots \times P(\mathsf{Te_{S5}})}_{n}$, for some $n \in \mathbb{N}$.

- By a *rule* we understand any subset $R \subseteq P(\mathsf{Te_{S5}})^n$ such that if $\langle X_1, \ldots, X_n \rangle \in R$, then:
  a. X_1 is t-inconsistent
  b. $X_1 \subset X_i$, for each $1 < i \leq n$.
- If $n \leq 2$, then each element R will be called an *ordered n-tuple* (pair, triple, etc., respectively).
- The first element of each n-tuple will be called an *input set* (*set of premises*), while its remaining elements *output sets* (*sets of conclusions)*.

In the case of **S5**, the rule definition differs from the rule definition for **TL** (definition 3.18) only in that it is specified on a different set of expressions and in a different nature of t-inconsistency. Beyond that, these definitions are structurally similar, because we are seeking to formulate general concepts. A set of tableau rules for the tableau system for **CPL** we describe, shall be introduced by means of another definition.

Definition 4.23 (Tableau rules for **S5**). Tableau rules for **S5** are the following rules:

$$R_{\wedge}: \frac{X \cup \{\langle (A \wedge B), i \rangle\}}{X \cup \{\langle (A \wedge B), i \rangle, \langle A, i \rangle, \langle B, i \rangle\}}$$

$$R_{\vee}: \frac{X \cup \{\langle (A \vee B), i \rangle\}}{X \cup \{\langle (A \vee B), i \rangle, \langle A, i \rangle\} \mid X \cup \{\langle (A \vee B), i \rangle, \langle B, i \rangle\}}$$

$$R_{\rightarrow}: \frac{X \cup \{\langle (A \rightarrow B), i \rangle\}}{X \cup \{\langle (A \rightarrow B), i \rangle, \langle \neg A, i \rangle\} \mid X \cup \{\langle (A \rightarrow B), i \rangle, \langle B, i \rangle\}}$$

$$R_{\leftrightarrow}: \frac{X \cup \{\langle (A \leftrightarrow B), i \rangle\}}{X \cup \{\langle (A \leftrightarrow B), i \rangle, \langle A, i \rangle, \langle B, i \rangle\} \mid X \cup \{\langle (A \leftrightarrow B), i \rangle, \langle \neg A, i \rangle, \langle \neg B, i \rangle\}}$$

$$R_{\neg\neg}: \frac{X \cup \{\langle \neg\neg A, i \rangle\}}{X \cup \{\langle \neg\neg A, i \rangle, \langle A, i \rangle\}}$$

$$R_{\neg\wedge}: \frac{X \cup \{\langle \neg(A \wedge B), i \rangle\}}{X \cup \{\langle \neg(A \wedge B), i \rangle, \langle \neg A, i \rangle\} \mid X \cup \{\langle \neg(A \wedge B), i \rangle, \langle \neg B, i \rangle\}}$$

$$R_{\neg\vee}: \frac{X \cup \{\langle\neg(A\vee B),i\rangle\}}{X \cup \{\langle\neg(A\vee B),i\rangle,\langle\neg A,i\rangle,\langle\neg B,i\rangle\}}$$

$$R_{\neg\rightarrow}: \frac{X \cup \{\langle\neg(A\rightarrow B),i\rangle\}}{X \cup \{\langle\neg(A\rightarrow B),i\rangle,\langle A,i\rangle,\langle\neg B,i\rangle\}}$$

$$R_{\neg\leftrightarrow}: \frac{X \cup \{\langle\neg(A\leftrightarrow B),i\rangle\}}{X \cup \{\langle\neg(A\leftrightarrow B),i\rangle,\langle\neg A,i\rangle,\langle B,i\rangle\} \mid X \cup \{\langle\neg(A\leftrightarrow B),i\rangle,\langle A,i\rangle,\langle\neg B,i\rangle\}}$$

$$R_{\neg\Box}: \frac{X \cup \{\langle\neg\Box A,i\rangle\}}{X \cup \{\langle\neg\Box A,i\rangle,\langle\Diamond\neg A,i\rangle\}}$$

$$R_{\neg\Diamond}: \frac{X \cup \{\langle\neg\Diamond A,i\rangle\}}{X \cup \{\langle\neg\Diamond A,i\rangle,\langle\Box\neg A,i\rangle\}}$$

$$R_{\Box}: \frac{X \cup \{\langle\Box A,i\rangle, irj\}}{X \cup \{\langle\Box A,i\rangle, irj, \langle A,j\rangle\}}$$

$$R_{\Diamond}: \frac{X \cup \{\langle\Diamond A,i\rangle\}}{X \cup \{\langle\Diamond A,i\rangle, irj, \langle A,j\rangle\}}, \text{ where:}$$

1. $j \notin *(X \cup \{\langle\Diamond A,i\rangle\})$
2. for any $k \in \mathbb{N}$, $\{irk, \langle A,k\rangle\} \nsubseteq X$.

$$R_r: \frac{X}{X \cup \{irj\}}, \text{ where } i,j \in *(X).$$

Set of tableau rules for **S5** will be defined as $\mathbf{R_{S5}}$.

Let us now devote a few words to discussing the described rules. Although in many ways they resemble the rules from set $\mathbf{R_{TL}}$, there are also some differences.

According to the definition of rule 4.22, the input sets of each rule are t-consistent. In addition, in each rule, the input set is basically contained in each output set. Again, there appears the two-premise rule — rule $R_{\Box}$.

On the other hand, similar to the case of rules R**i** and R**o** for **TL**, we have a rule with limitations on introducing new expressions. We mean rule $R_{\Diamond}$. This rule says that the expressions introduced are to have a new index (condition 1), moreover, the input set must not contain expressions similar to the one entered, beginning with the index appearing in the expression with $\Diamond$ (condition 2).

The semantic intuitions on which condition 1 is based are as follows. The object denoted by a new index is also to be new and its relationship with other objects that are denoted by the other indices is to remain unresolved.

Condition 2 prevents unnecessary expressions from being entered in the proof. In practice, it also prevents the proof from being unnecessarily extended indefinitely — so it prevents the creation of infinite branches when this is not a consequence of logic itself, but of the wrong definition of tableau system (example 4.24).

The last rule is a rule that corresponds to universal relation in the models. With rule R_r, each two indices that appear in the expressions found in the proof may cause the addition to the proof of another expression that contains those indices. These indices do not have to be different, of course.

Example 4.24. Consider the following set of expressions $X \cup \{\langle \Diamond p, 1\rangle\}$ and rule $R_\Diamond$ without condition 2. The use of rule $R_\Diamond$ without condition 2 can result in infinite branches by entering expressions with new indices.

$$
\begin{array}{rcl}
R_\Diamond\ X_1 & = & X \cup \{\langle \Diamond p, 1\rangle\} \\
 & | & \\
R_\Diamond\ X_2 & = & X_1 \cup \{1r2, \langle p, 2\rangle\} \\
 & | & \\
R_\Diamond\ X_3 & = & X_2 \cup \{1r3, \langle p, 3\rangle\} \\
 & | & \\
R_\Diamond\ X_4 & = & X_3 \cup \{1r4, \langle p, 4\rangle\} \\
 & | & \\
 & \ldots &
\end{array}
$$

The example presented above could be part of some kind of proof in which we could still apply rule $R_\Diamond$ to each new set without condition 2, utilizing the fact that it contains expression $\langle \Diamond A, 1\rangle$.

In a standard, informal and intuitive approach to the logic within a tableau approach[2], we can find an equivalent of rule $R_\Diamond$ — below, we denote it in a conventional form:

$\Diamond A, i$
$\downarrow$
irj
A, j, where j is new in the branch.

However, this formulation of the rule causes various problems. The basic problem — a fundamental one, we can say — is that the rule refers to the concept of

2 For instance, in the study by G. Priest, *Introduction to Non-classical Logic* [23].

branch, although the concept of branch is connected with the concept of set of rules — in our approach a branch is determined by the application of rules from the output set of rules. So, this way of defining a rule has the features of circularity — we define a rule by reference to a branch, whereas the definition of branch requires the definition of set of rules.

In addition, this formulation of rule $R_{\Diamond}$ seems too weak. In literature, it is only the comments to the rule that prohibit its application to the same expression more than once[3].

In the view presented in the book, rule $R_{\Diamond}$ formally requires a new index when applied (condition 1 allows only such pairs of sets), but also blocks the creation of *catch-22 situation* of applications (due to condition 2). In the approach we offer, we formally subsume the fact that *index must be new* and *the rule can only be applied to the same formula once*. Besides, the latter formulation is not precise either; in some cases condition 2 does not allow us to apply rule $R_{\Diamond}$ to given formula even once (example 4.25).

Example 4.25. Consider the following set of expressions $X \cup \{\langle \Diamond p, 1\rangle, \langle \Diamond (p \wedge q), 1\rangle\}$ and rule $R_{\Diamond}$ with condition 2.

$$\begin{array}{rl} R_{\Diamond} & X_1 = X \cup \{\langle \Diamond p, 1\rangle, \langle \Diamond (p \wedge q), 1\rangle\} \\ & \qquad | \\ R_{\wedge} & X_2 = X_1 \cup \{1r2, \langle (p \wedge q), 2\rangle\} \\ & \qquad | \\ & X_3 = X_2 \cup \{\langle p, 2\rangle, \langle q, 2\rangle\} \end{array}$$

To set X_1 we applied rule $R_{\Diamond}$, drawing conclusions from expression $\langle \Diamond (p \wedge q), 1\rangle$. Next, to set X_2 we applied rule $R_{\wedge}$ by decomposing expression $\langle (p \wedge q), 2\rangle$. However, to set X_3 we cannot anymore apply rule $R_{\Diamond}$ in order to decompose expression $\langle \Diamond p, 1\rangle$, since X_3 contains expressions $1r2$ and $\langle p, 2\rangle$. So, to expression $\langle \Diamond p, 1\rangle$ we did not apply rule $R_{\Diamond}$ even once.

However, as noted before, in modal logic, even application of conditions blocking the unnecessary use of rules may not prevent the emergence of infinite branches, which will be proven later, having defined the concept of modal branch.

Remark 4.26. We can consider alternative sets of rules for the construction of a tableau system for **S5**. The following rules would help (on their right side we put the condition that fulfils the relation in the model).

3 This is why G. Priest notes: *in the rule for* $\diamond$ *(...) the number j must be new*, *not mentioning that this rule can be applied only once for one formula* ([23], s. 25).

(Reflexivity)

$$R_{ref}: \frac{X \cup \{\langle A,i \rangle\}}{X \cup \{\langle A,i \rangle, iri\}} \qquad \forall_{w_1 \in W}\, w_1 R w_1$$

(Symmetry)

$$R_{sym}: \frac{X \cup \{irj\}}{X \cup \{irj, jri\}} \qquad \forall_{w_1,w_2 \in W}\, (w_1 R w_2 \Rightarrow w_2 R w_1)$$

(Transitivity)

$$R_{trans}: \frac{X \cup \{irj, jrk\}}{X \cup \{irj, jrk, irk\}} \qquad \forall_{w_1,w_2,w_3 \in W}\, (w_1 R w_2 \,\&\, w_2 R w_3 \Rightarrow w_1 R w_3)$$

The above rules correspond to the properties of equivalence relations: reflexivity, transitivity and symmetry. Fact 4.11 states that a class of models with universal relation and a class of models with equivalence relation define exactly the same logic. Therefore, an alternative set of tableau rules could be defined through the above three rules: $(\mathbf{R_{S5}} \setminus R_r) \cup \{R_{ref}, R_{sym}, R_{trans}\}$.

Although both sets of rules seem to be interesting, for the reasons we have already described, we will not investigate the second one — for we intend to provide a generalisation further in the book which will also include this approach. So, now we will focus on the tableau system determined by set of rules $\mathbf{R_{S5}}$. The example provided indicates that the general definition of rule 4.22 makes sense. There may exist many sets of tableau rules that potentially — to determine this always requires a proof — define the same logic.

The adopted set of tableau rules — in our case set $\mathbf{R_{S5}}$ — specifies the content of the range of successive concepts of the tableau system. Formally, the concepts we will describe will be analogous to those from the previous chapter. However, each of them will depend on set of tableau rules $\mathbf{R_{S5}}$.

4.3.2 Branches for S5

With a fixed set of tableau rules, we can proceed to the concept of branch. Branches are such sequences of sets that each two adjacent elements constitute an input set and an output set of some n-tuple that belongs to the set of tableau rules. Branches are therefore setwise objects consisting of sets. Let us now proceed to the formal definition of branch in the tableau system for **S5** and derived concepts. The individual concepts will be similar to those from the previous chapters, and all of them will be defined on the currently adopted set $\mathsf{Te}_{\mathbf{S5}}$.

Definition 4.27 (Branch)**.** Let $K = \mathbb{N}$ or $K = \{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$. Let X be any set of expressions. *A branch* (or *a branch beginning with* X) will be called any sequence $\phi : K \longrightarrow P(\mathsf{Te}_{\mathbf{S5}})$ that meets the following conditions:

1. $\phi(1) = X$
2. for any $i \in K$, if $i+1 \in K$, then there exists such rule $R \in \mathbf{R_{S5}}$ and such n-tuple $\langle Y_1, \ldots, Y_n \rangle \in R$, that $\phi(i) = Y_1$ and $\phi(i+1) = Y_k$, for certain $1 < k \leq n$.

Having two branches ϕ, ψ such that $\phi \subset \psi$ we shall state that:

- ϕ is a sub-branch of ψ
- ψ is a super-branch of ϕ.

Denotation 4.28. From now on — when speaking of branches — for convenience, we will use the following notations or designations:

1. $X_1, \ldots, X_n$, where $n \geq 1$
2. $\langle X_1, \ldots, X_n \rangle$, where $n \geq 1$
3. abbreviations: ϕ_K, where K is a domain ϕ, i.e. $\phi : K \longrightarrow P(\mathsf{Te_{S5}})$
4. or — to denote branches — small Greek letters: ϕ, ψ, etc.

The sets of branches, in turn, we shall denote with capital Greek letters: Φ, Ψ, etc. Furthermore, the domain cardinality of a given branch K we shall sometimes call a *length* of that branch.

The concept of branch depends on the initial set of tableau rules. In the case under consideration, the branch structure is based on the rules from set $\mathbf{R_{S5}}$. Further described tableau concepts also depend on that set of rules. In harmony with the adopted convention, we will not entangle the notations since alls the subsequent concepts depend on set $\mathbf{R_{S5}}$.

By definition of rules 4.22, through the fact that they are defined by proper inclusion of the input set in each of the output sets in any n-tuple, we get a conclusion.

Corollary 4.29. *Each branch is an injective sequence.*

4.3.3 Closed and open branches

Unlike usual, we will first take up a certain type of maximal branches, and only then we will introduce the concept of maximal branch. The reason for this is that because of the emergence of infinite branches beginning with finite sets of expressions, we will have to extend the concept of maximal branch. The concept of a closed branch will be useful for the extension, so we start with it. In addition, the set of closed branches is completed by the set of open branches, so we will also introduce the concept of open branch.

As we remember, intuitively a branch is closed when we get a t-inconsistent set having decomposed expressions by rules.

Definition 4.30 (Closed/open branch)**.** Branch $\phi : K \longrightarrow P(\mathsf{Te}_{\mathbf{S5}})$ will be called *closed* iff $\phi(i)$ is a t-inconsistent set for some $i \in K$. Branch ϕ will be called *open* iff it is not closed.

From the above definition, the definition of tableau rules for **S5** 4.23 and the definition of branch 4.27, we get the fact.

Proposition 4.31. *If branch* $\phi : K \longrightarrow P(\mathsf{Te}_{\mathbf{S5}})$ *is closed, then* $|K| \in \mathbb{N}$ *and* $\phi(|K|)$ *is a t-inconsistent set.*

In the case of a closed branch, the t-inconsistent sequence element is the last element because no rule can be applied to it anymore to extend the branch. For the rules are defined in such a way that they cannot be applied to t-inconsistent sets.

4.3.4 Maximal branches

We will commence the issue of maximal branches for the tableau system for **S5** with an initial and non-proper concept of maximal branch. By analogy to the definitions of maximal branch in tableau systems for **CPL** and **TL** (definitions 2.30 and 3.26), we could adopt the following definition.

Definition 4.32 (Maximal branch — variant one)**.** Let $\phi : K \longrightarrow P(\mathsf{Te}_{\mathbf{S5}})$ be a branch. We shall state that ϕ is *maximal* iff

1. $K = \{1, 2, 3, \ldots, n\}$, for some $n \in \mathbb{N}$
2. there is no branch ψ such that $\phi \subset \psi$.

Unfortunately, there are cases where the first set is finite, but this does not guarantee the finiteness of branch. So, regardless of either we used all the rules from set $\mathbf{R}_{\mathbf{S5}}$ which could have been applied to some expressions appearing in the branch, or we still have expressions to which no rules were applied, the branch can be infinite.

Example 4.33. Consider an example of set $\{\langle \neg(\Diamond p \rightarrow \Diamond \Box p), 1\rangle\}$. It is a finite set. Below, we describe an infinite branch beginning with set X and constructed by applying the rules from set of tableau rules $\mathbf{R}_{\mathbf{S5}}$.

$$
\begin{array}{ll}
R_{\neg} & X_1 = \{\langle\neg(\Diamond p \rightarrow \Diamond\Box p), 1\rangle\} \\
& \quad | \\
R_{\Diamond} & X_2 = X_1 \cup \{\langle\Diamond p, 1\rangle, \langle\neg\Diamond\Box p, 1\rangle\} \\
& \quad | \\
R_{\neg\Diamond} & X_3 = X_2 \cup \{1r2, \langle p, 2\rangle\} \\
& \quad | \\
R_{\Box} & X_4 = X_3 \cup \{\langle\Box\neg\Box p, 1\rangle\} \\
& \quad | \\
R_{\neg\Box} & X_5 = X_4 \cup \{\langle\neg\Box p, 2\rangle\} \\
& \quad | \\
R_{\Diamond} & X_6 = X_5 \cup \{\langle\Diamond\neg p, 2\rangle\} \\
& \quad | \\
R_{r} & X_7 = X_6 \cup \{2r3, \langle\neg p, 3\rangle\} \\
& \quad | \\
R_{\Box} & X_8 = X_7 \cup \{1r3\} \\
& \quad | \\
R_{\neg\Box} & X_9 = X_8 \cup \{\langle\neg\Box p, 3\rangle\} \\
& \quad | \\
R_{\Diamond} & X_{10} = X_9 \cup \{\langle\Diamond\neg p, 3\rangle\} \\
& \quad | \\
& \quad \ldots
\end{array}
$$

The branch of infinite length is obtained by applying by turns the following rules $R_{\Diamond}$, R_r, $R_{\Box}$, $R_{\neg\Box}$ to set X_{10} and to each subsequent set that is created following the application of the given sequence of rules.

The definition of maximal branch in the given version specifies that the maximal branch is finite and there exists no super-branch for it. This definition is good for those systems where applying tableau rules to finite sets of expressions always produces finitely long branches. In these cases, using the tableau rules for a given system, we can decompose all the initial expressions from a finite set of tableau expressions in a finite number of steps. It is therefore not a good definition for modal logic because of the following fact.

Proposition 4.34. *There exists such finite set of expressions X that a branch beginning with X is infinite.*

Proof. Example 4.33. □

It would appear that removal of the first condition from definition 4.32 will improve the situation. Removal of the first condition would mean that the branch does not have to be of finite length. Its maximality would be based on the fact that there exists no super-branch of it. However, the lack of a super-branch for the branch beginning with a finite set does not mean that all the tableau rules that could be used to construct the branch were actually used. Even in example 4.33, in individual sequences, we did not make use of all the possibilities of using rule R_r.

Note that by adding to set $X = \{\langle\neg(\Diamond p \to \Diamond\Box p), 1\rangle\}$ any new expression — e.g. $\langle(q \wedge \neg q), 1\rangle$, we get a finite set again. For this set, there exists a branch of infinite length in which the rule for this new expression has not been applied in any of the subsets (example 4.35).

Therefore, even though the branch may satisfy the second condition from definition 4.32 — there is no branch to be contained in it — it does not, however, draw all the possible conclusions from the expressions that belong to the elements of this branch, including the initial set.

Example 4.35. Consider an example of set of expressions $X = \{\langle\neg(\Diamond p \to \Diamond\Box p), 1\rangle, \langle(q \wedge \neg q), 1\rangle\}$. It is a finite set. Below, we describe an infinite branch beginning with set X and constructed by applying the rules from set of tableau rules $\mathbf{R_{S5}}$. This branch is similar to the one in example 4.33, except that each of its elements contains expression $\langle(q \wedge \neg q), 1\rangle$ from which in none of the branch elements conclusions have been drawn applying rule $R_\wedge$.

$$
\begin{array}{ll}
R_\neg & X_1 = \{\langle\neg(\Diamond p \to \Diamond\Box p), 1\rangle, \langle(q \wedge \neg q), 1\rangle\} \\
 & \quad | \\
R_\Diamond & X_2 = X_1 \cup \{\langle\Diamond p, 1\rangle, \langle\neg\Diamond\Box p, 1\rangle\} \\
 & \quad | \\
R_{\neg\Diamond} & X_3 = X_2 \cup \{1r2, \langle p, 2\rangle\} \\
 & \quad | \\
R_\Box & X_4 = X_3 \cup \{\langle\Box\neg\Box p, 1\rangle\} \\
 & \quad | \\
R_{\neg\Box} & X_5 = X_4 \cup \{\langle\neg\Box p, 2\rangle\} \\
 & \quad | \\
R_\Diamond & X_6 = X_5 \cup \{\langle\Diamond\neg p, 2\rangle\} \\
 & \quad | \\
R_r & X_7 = X_6 \cup \{2r3, \langle\neg p, 3\rangle\} \\
 & \quad | \\
R_\Box & X_8 = X_7 \cup \{1r3\} \\
 & \quad | \\
R_{\neg\Box} & X_9 = X_8 \cup \{\langle\neg\Box p, 3\rangle\} \\
 & \quad | \\
R_\Diamond & X_{10} = X_9 \cup \{\langle\Diamond\neg p, 3\rangle\} \\
 & \quad | \\
 & \quad \ldots
\end{array}
$$

A branch of infinite length is obtained by applying by turns the following rules $R_\Diamond$, R_r, $R_\Box$, $R_{\neg\Box}$ to set X_{10} and to each subsequent set that is created following the application of the given sequence of rules. Whereas, in no step we apply rule $R_\wedge$.

Before we can introduce a somewhat more general and ultimate definition of maximal branch, we still need a few auxiliary concepts.

Definition 4.36 (Core of rule). Let rule $R \in \mathbf{R_{S5}}$ and $n \in \mathbb{N}$. Let $\langle X_1, \ldots, X_n \rangle \in R$ and $\langle Y_1, \ldots, Y_n \rangle \in R$. We shall state that set $\langle Y_1, \ldots, Y_n \rangle \in R$ is a *core of rule R in set* $\langle X_1 \ldots, X_n \rangle$ iff

1. $Y_1 \subseteq X_1$
2. there exists no such proper subset $U_1 \subset Y_1$ that for some n-tuple $\langle U_1, \ldots, U_n \rangle \in R$
3. for any $1 < i \leq n$, $Y_i = Y_1 \cup (X_i \setminus X_1)$.

Remark 4.37. Note that in the case of tableau rules from set $R \in \mathbf{R_{S5}}$:

- the cores of rule for given n-tuple are expressions that play important roles at given stage of the tableau proof — most often, the core input set is one-element set
- in a special case, the core input set for the rule for given n-tuple is two-element set of expressions — it is the case, for instance, in the event of rule $R_{\Box}$.

Even though the rules have been defined on sets, the concept of core of rule in set indicates an essential element or elements which enable drawing conclusions through the use of rule.

The above concepts result in the following conclusion.

Corollary 4.38. *Let rule $R \in \boldsymbol{R_{S5}}$, $n \in \mathbb{N}$, and let $\langle X_1, \ldots, X_n \rangle \in R$. Then, there exists such n-tuple $\langle Y_1, \ldots, Y_n \rangle \in R$ that $\langle Y_1, \ldots, Y_n \rangle$ is the core of rule R in set $\langle X_1, \ldots, X_n \rangle$.*

Proof. By definition of tableau rules 4.23 and by definition of core of rule 4.36. □

Now, we can proceed to the concept of strong similarity between the sets of expressions. It is a speciality of the concept of similarity (definition 4.16). We will need the concept of strong similarity for the definition of maximal branch.

Definition 4.39 (Strong similarity). Let rule $R \in \mathbf{R_{S5}}$ and let $\langle X_1, \ldots, X_n \rangle \in R$, for some $n \in \mathbb{N}$. On any set of expressions $W \subseteq \mathsf{Te}_{\mathbf{S5}}$, we will state that it is *strongly similar to* set X_i, where $1 < i \leq n$, iff

1. W is similar to X_i
2. for certain n-tuple $\langle Y_1, \ldots, Y_n \rangle$, which is the core of rule R in set $\langle X_1, \ldots, X_n \rangle$, and for certain $W' \subseteq W$, the following conditions are met:
   a. $Y_1 \subseteq W'$
   b. W' is similar to $Y_1 \cup (X_i \setminus X_1)$.

Having adopted the concept of strong similarity, we can proceed to the concept of maximal branch in the final version.

Definition 4.40 (Maximal branch)**.** Let $\phi : K \longrightarrow P(\mathsf{Te}_{\mathbf{S5}})$ be a branch. We shall state that ϕ is *maximal* iff it meets one of the below conditions:

1. ϕ is closed
2. for any rule $R \in \mathbf{R}_{\mathbf{S5}}$, any $n \in \mathbb{N}$ and any n-tuple $\langle X_1, \ldots, X_n \rangle \in R$, if $\phi(k) = X_1$, for certain $k \in K$, then for some $j \in K$, there exist $\phi(j)$ and such set of expressions $W \subseteq \mathsf{Te}_{\mathbf{S5}}$ that for some $1 < i \leq n$, W is strongly similar to X_i and $W \subseteq \phi(j)$.

Remark 4.41. According to the above definition, a maximal branch is closed or, in a sense, closed under application of rules (both conditions do not necessarily have to be mutually exclusive). Closure under rules means that if a branch is not closed and it was possible to apply some rule to one of its elements, then some of the branch elements includes a set strongly similar to the one that could have been a result of application of that rule. There are two things worth clarifying.

In the definition, we mention a strongly similar set W because the application of rule $R_{\Diamond}$, in subsequent stages, can result in different sets than the ones we would have gotten by applying this rule earlier.

Set W is to be contained in one of the elements of branch $\phi(j)$, and not necessarily be identical to it, since the rule could have been applied to set $\phi(j-1)$ which can be a proper superset of set X.

Therefore, maximal branches can be either finite or infinite. Of course, if a branch is maximal in terms of the first definition 4.32, that means it is maximal in terms of the definition we have adopted — 4.40.

Corollary 4.42. *Each branch which is maximal in terms of definition 4.32 is maximal in terms of definition 4.40.*

Proof. Take any branch ϕ maximal in terms of definition 4.32 and assume that ϕ is not closed. If it does not meet the second condition of definition 4.40, then since ϕ is finite — the first condition of definition 4.32, so there exists branch ψ such that $\phi \subset \psi$ which obviously contradicts the second condition of definition 4.32. □

We know, therefore, that the concept of maximal branch as used so far, expressed in definition 4.32 is a special case of the concept defined by the definition of maximal branch 4.40. The latter definition is therefore taken as a model definition for the theory of tableau systems.

Important cases for the theory of tableaux that meet the definition of maximal branch used in the chapters on the tableau system for **CPL** and **TL**, also meet definition 4.40. Important cases, both for the branch consequence relation and the tableau, are those where the initial set — a set of tableau expressions or formulas — is finite. Therefore, since we know that from finite sets in the tableau system for **CPL** and in the tableau system for **TL** only emerge finite branches (see fact 2.34 and fact 3.31), so we can state that definitions like 4.40 by virtue of conclusion 4.42 — we mean *like* since even though obviously, constructively the same idea stands behind them, in both cases we face different rules and different sets of expressions — are appropriate for the system with the property of a finite branch.

From definitions 4.30 and 4.40, we get a self-evident conclusion.

Corollary 4.43. *Each closed branch is maximal.*

Therefore, sometimes when constructing a tableau proof using tableau tools for logic **S5**, when we seek maximal branches, we can deal with infinite branches. It is understandable that those branches cannot be described with anything different than a scheme. In general, branches, as part of a tableau proof, are not sequences put on paper, but sequences of abstract objects that we can or cannot say are or are not maximal.

4.3.5 Relation of branch consequence

As in the case of previous tableau systems, we will also define the concept of branch consequence for the presently described system using the following terms: branch, maximal branch and closed branch.

Definition 4.44 (Branch consequence **S5**). Let $X \subseteq \mathsf{For}_{\mathbf{S5}}$ and $A \in \mathsf{For}_{\mathbf{S5}}$. Formula A is a *branch consequence* of X (for short: $X \rhd A$) iff there exists such finite set $Y \subseteq X$ and index $i \in \mathbb{N}$ that each maximal branch beginning with set $\{\langle B, i\rangle : B \in Y \cup \{\neg A\}\}$ is closed.

Denotation 4.45. For any set of formulas X and any formula A notation $X \not\rhd A$ will mean that it is not the case that $X \rhd A$.

The above concept of branch consequence relation differs from the analogous concept for tableau system for **CPL** and **TL** in the fact that (obviously, apart from defining on a different language — different set of expressions) when determining whether given pair belongs to the relation of branch consequence we can encounter a problem of branches that are maximal and infinite alike.

4.4 Tableaux for S5

It is usually inconvenient, or even infeasible, to investigate whether a pair belongs to the relation of branch consequence. As we remember, in the approach presented in the book, it is the construction of tableau that is supposed to solve this problem.

When defining a tableau for logic **S5**, we will readdress the auxiliary concept of maximality in a set of branches. It is necessary because, unlike the case of tableau system for **TL**, and like the case of **CPL**, a tableau can contain many branches. So for the sake of order, we must avoid a situation in which sub-branches belong to the same tableau as their super-branches.

Definition 4.46 (Maximal branch in the set of branches)**.** Let Φ be a set of branches and let branch $\psi \in \Phi$. We shall state that ψ is *maximal in set* Φ (for short: Φ–*maximal*) iff there is no such branch $\phi \in \Phi$ that $\psi \subset \phi$.

We can now move on to the concept of tableau. Tableau is a special and non-empty set of branches that *i*) begin with the same set of expressions and *ii*) each branching requires a tableau rule that allows for such branching, and what is more *iii*) each branch that belongs to the tableau must be maximal in this set.

Definition 4.47 (Tableau)**.** Let $X \subseteq \mathsf{For}_{\mathbf{S5}}$, $A \in \mathsf{For}_{\mathbf{S5}}$ and Φ be a set of branches. Ordered triple $\langle X, A, \Phi \rangle$ will be called *a tableau for* $\langle X, A \rangle$ (or for short: *tableau*) iff the below conditions are met:

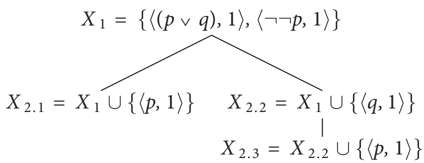
1. Φ is a non-empty subset of set of branches beginning with $\{\langle B, i \rangle : B \in X \cup \{\neg A\}\}$, for some index $i \in \mathbb{N}$ (i.e. if $\psi \in \Phi$, then $\psi(1) = \{\langle B, i \rangle : B \in X \cup \{\neg A\}\}$)
2. each branch contained in Φ is Φ-maximal
3. for any $n, i \in \mathbb{N}$ and any branches $\psi_1, \ldots, \psi_n \in \Phi$, if:
   - i and $i+1$ belong to domains of functions $\psi_1, \ldots, \psi_n$
   - for any $1 < k \leq n$ and any $o \leq i$, $\psi_1(o) = \psi_k(o)$

   then there exists such rule $R \in \mathbf{R_{S5}}$ and such ordered m-tuple $\langle Y_1, \ldots, Y_m \rangle \in R$, where $1 < m \leq 3$ that for any $1 \leq k \leq n$:
   - $\psi_k(i) = Y_1$
   - and there exists such $1 < l \leq m$ that $\psi_k(i+1) = Y_l$.

Again, the concept of tableaux has been defined in such a way that tableaux can also begin with infinite sets. Practicably, however, the construction of tableau is to show that a given formula is a branch consequence of given finite set of premises — according to the definition of branch consequence 4.44. To this end, we must construct tableaux containing all elements that are sufficient to solve the problem. Such tableaux are called complete tableaux. Before that, we will consisder the problem of redundant branches.

The definition of redundant branches resembles a similar definition, worded for **CPL**, while the difference, as usual, boils down to the fact that its subject are branches composed of other objects. Let us consider an example analogous to the one we discussed in the chapter devoted to the classical logic (example 4.48).

Example 4.48. Consider set of expressions $\{\langle p \vee q, 1\rangle, \langle \neg\neg p, 1\rangle\}$. Through the use of rule $R_{\vee}$, we get two branches, and then to set $X_{2.2}$ we apply rule $R_{\neg\neg}$. Then we get the following branches.



In the light of definition of tableau 4.47, the set of these two branches is a tableau for pair $\langle\{p \vee q\}, \neg p\rangle$.

However, from the viewpoint of a tableau complexity, the branch on the right seems unnecessary. This is because if we fail to get t-inconsistent set in the right-hand branch, we will also fail to get t-inconsistent set in the left-hand branch. Therefore, the branch on the right seems superfluous, and since it brings nothing important, it can be conventionally called redundant.

Let us also repeat that such branchings do not form a formal obstacle. We suggest their bypassing merely for the sake of the economy of a construction. Therefore, further concepts will be defined in a similar way as in Chapter Two, i.e. so that the tableau with or without redundant branches can be considered a complete tableau. Practically, we know that when we try to write or draw out a tableau proof, we endeavour to take account of all possibilities. If required, however, the redundant branches can be bypassed. Let us now update the concept of redundant variant of branch for the current context.

Definition 4.49 (Redundant variant of branch)**.** Let ϕ and ψ be such branches that for some numbers i and $i+1$ that belong to their domains, it is the case that for any $j \leq i$, $\phi(j) = \psi(j)$, but $\phi(i+1) \neq \psi(i+1)$. We shall state that branch ψ is a *redundant variant* of branch ϕ iff:

- there exists such rule $R \in \mathbf{R_{S5}}$ and such pair $\langle X, Y\rangle \in R$ that $X = \phi(i)$ and $Y = \phi(i+1)$
- there exists such rule $R \in \mathbf{R_{S5}}$ and such triple $\langle X, W, Z\rangle \in R$ that $X = \phi(i)$ and:

1. $W = \phi(i+1)$ and $Z = \psi(i+1)$
   or
2. $Z = \phi(i+1)$ and $W = \psi(i+1)$.

Let Φ, Ψ be sets of branches and $\Phi \subset \Psi$. We shall state that Ψ is a *redundant superset* Φ iff for any branch $\psi \in \Psi \setminus \Phi$ there exists such branch $\phi \in \Phi$ that ψ is a redundant variant of ϕ.

Having adopted the concept of redundant superset of branches, we can proceed to the definition of complete tableau.

Definition 4.50 (Complete tableau)**.** Let $\langle X, A, \Phi \rangle$ be a tableau. We shall state that $\langle X, A, \Phi \rangle$ is *complete* iff:

1. each branch contained in Φ is maximal
2. any set of branches Ψ such that:
   a. $\Phi \subset \Psi$
   b. $\langle X, A, \Psi \rangle$ is a tableau

   is a redundant superset of Φ.

We shall state that a tableau is *incomplete* iff it is not complete.

A complete tableau contains such set of branches that it is no longer possible to add any new branches to it, without causing the ordered triple to cease to be a tableau, or there appears a redundant variant of some branch. As we already know, in a complete tableau, all branches are maximal, not only the maximal ones in a given set. Thus, a complete tableau contains such set of maximal branches that any of its supersets does not anymore produce a tableau, or it features at least one redundant variant of some branch that already earlier belonged to the tableau.

When constructing a complete tableau, we can face a situation in which all the branches are closed, meaning each branch ends with a set that is t-inconsistent. Such a tableau will be called a closed tableau.

Definition 4.51 (Closed/open tableau)**.** Let $\langle X, A, \Phi \rangle$ be a tableau. We shall state that $\langle X, A, \Phi \rangle$ is *closed* iff the below conditions are met:

1. $\langle X, A, \Phi \rangle$ is a complete tableau
2. each branch contained in Φ is closed.

We shall state that a tableau is *open* iff it is not closed.

By the above definition of closed tableau, we get another conclusion on the relation of closed tableaux with complete ones.

Corollary 4.52. *Each closed tableau is a complete tableau.*

4.5 Theorem on the completeness of the tableau system for S5

Further, we will show that the concept of tableau is significantly helpful in determining the occurrence of relation $\rhd$, and that the existence of a closed tableau is equivalent to the occurrence of semantic consequence $\models$. But before we move on to these problems, we must introduce a few definitions and determine a few facts.

First, we will show that for any $X \subseteq \mathsf{For}_{\mathbf{S5}}$, $A \in \mathsf{For}_{\mathbf{S5}}$, if $X \models A$, then $X \rhd A$. Let us begin with the definition of closure under tableau rules.

Definition 4.53 (Closure under tableau rules)**.** Let $X \subseteq \mathsf{Te}_{\mathbf{S5}}$. We shall state that set $Y \subseteq \mathsf{Te}_{\mathbf{S5}}$ is a *closure of set X under tableau rules* iff Y is a set that meets the following conditions:

- $X \subseteq Y$
- for any rule $R \in \mathbf{R_{S5}}$ and any n-tuple $\langle Z_1, Z_2, \ldots, Z_n \rangle \in R$, where $n \in \mathbb{N}$, if $X \subseteq Z_1 \subseteq Y$, then $Z_j \subseteq Y$, for some $2 \leq j \leq n$.

If set Y is a closure of set X under tableau rules, then it will be denoted as X^Y. Sometimes, on set Y we will simply state that it is a *closure*.

Obviously, each set of expressions X has its closure Y such that $X \subseteq Y \subseteq \mathsf{Te}_{\mathbf{S5}}$. Some sets can have more that one closure. Note that by definition of branch 4.27, the following fact occurs.

Proposition 4.54. *Each closure under tableau rules is a branch of length one.*

Making use of the concept of closure, we will now show a relationship between the existence of maximal and open branches originating from finite subsets of some set of expressions and the existence of closure of that set which is an open and maximal branch.

Lemma 4.55 (On the existence of maximal and open branch)**.** *Let $X \subseteq \mathsf{For}_{\mathbf{S5}}$ and $i \in \mathbb{N}$. If for any finite subset $Y \subseteq X$ there exists an open and maximal branch beginning with set of expressions $Y^i = \{\langle A, i \rangle : A \in Y\}$, then there exists such closure Z of set of expressions $X^i = \{\langle A, i \rangle : A \in X\}$ under tableau rules that Z is an open and maximal branch.*

Proof. Take any $X \subseteq \mathsf{For}_{\mathbf{S5}}$ and $i \in \mathbb{N}$. Next, assume that $(*)$ for any finite subset $Y \subseteq X$ there exists open and maximal branch beginning with set of expressions $Y^i = \{\langle A, i \rangle : A \in Y\}$.

Now, we take the set of all maximal and open branches beginning with set $Y^i = \{\langle A, i \rangle : A \in Y\}$, for any finite subset $Y \subseteq X$. The set will be denoted as $\mathbf{X}$.

Next, we will define set $\overline{\mathbf{X}}$ with the following conditions:

1. $\overline{\mathbf{X}} \subseteq \mathbf{X}$
2. for any two branches ϕ and ψ contained in $\mathbf{X}$, if there exist such two numbers $i, k \in \mathbb{N}$ that $\phi(i) \cup \psi(k)$ is a t-inconsistent set, then $\phi \notin \overline{\mathbf{X}}$ or $\psi \notin \overline{\mathbf{X}}$
3. $\overline{\mathbf{X}}$ is a maximal set among subsets $\mathbf{X}$ that meet conditions 1 and 2.

There exists at least one such set $\overline{\mathbf{X}}$ that $\overline{\mathbf{X}} \subseteq \mathbf{X}$. Take one such set $\overline{\mathbf{X}}$ and denote it as $\overline{X}$.

Consider set $\bigcup\{\phi(1) : \phi \in \overline{X}\}$. Note that $(**)$ $X^i \subseteq \bigcup\{\phi(1) : \phi \in \overline{X}\}$. If that was not the case, there would exist such $x \in X^i$ that $x \notin \bigcup\{\phi(1) : \phi \in \overline{X}\}$ and for each such branch $\psi \in \mathbf{X}$ that $x \in \psi(1)$, $\psi(1) \subseteq X^i$ and $\psi(1)$ is a finite set, it would be the case that $\psi \notin \overline{X}$. And then, for some finite subset $Y^i \subseteq X^i$ there would exist no maximal and open branch beginning with set $Y^i \cup \{x\}$ which contradicts assumption $(*)$.

Now, using condition:

$$U \in \overline{\overline{X}} \text{ iff there exists such branch } \phi \text{ that } \phi \in \overline{X} \text{ and } U = \bigcup\phi$$

we define set $\overline{\overline{X}}$. Further, we define set $Z = \bigcup\overline{\overline{X}}$.

We claim that set Z is a closure of set of expressions $X^i = \{\langle A, i\rangle : A \in X\}$ under tableau rules, according to definition 4.53, and that Z is an open and maximal branch.

First, we will show that Z is a closure of set of expressions $X^i = \{\langle A, i\rangle : A \in X\}$. Thus, we will show that Z meets conditions of closure, according to definition 4.53.

Note that $X^i \subseteq Z$, since by $(**)$ $X^i \subseteq \bigcup\{\phi(1) : \phi \in \overline{X}\}$, and by construction of set Z, $\bigcup\{\phi(1) : \phi \in \overline{X}\} \subseteq Z$.

Now, take any rule $R \in \mathbf{R_{S5}}$ and any n-tuple $\langle U_1, \ldots, U_n\rangle \in R$, for some $n \in \mathbb{N}$, and assume that $X^i \subseteq U_1 \subseteq Z$. From the definition of tableau rules for **S5** 4.23, it follows that there exists such n-tuple $\langle U_1', \ldots, U_n'\rangle \in R$ that:

- for any $1 \leq j \leq n$, U_j' is a minimal finite set such that if U_j is not such a minimal finite set that $\langle U_1, \ldots, U_n\rangle \in R$, then $U_j' \subset U_j$
- for any $1 < j \leq n$, $U_j \setminus U_1 = U_j' \setminus U_1'$.

Therefore, assuming that $U_1' \subseteq Z$, we must show that for some $1 < l \leq n$, $U_l' \subseteq Z$, since $U_l' \cup U_1 = U_l$. Since for finite set of expressions U_1', it is the case that $U_1' \subseteq Z$, there exists such finite number of branches $\phi_1, \phi_2, \ldots, \phi_o$ in set $\overline{X}$ that for some $k \in \mathbb{N}$, $U_1' \subseteq \phi_1(k) \cup \phi_2(k) \cup \cdots \cup \phi_o(k)$. So, set $\overline{X}$ contains branch ψ such that $\psi(1) = \phi_1(1) \cup \phi_2(1) \cup \cdots \cup \phi_o(1)$ and $U_1' \subseteq \psi(m)$, for certain $m \in \mathbb{N}$, since ψ is a maximal branch and $\phi_1(k) \cup \phi_2(k) \cup \cdots \cup \phi_o(k)$ is a t-consistent set. But, from the fact that ψ is a maximal branch and from definition of maximal branch 4.40, it follows that for certain $1 < l \leq n$, $U'_l \subseteq \bigcup \psi$, and thus $U'_l \subseteq Z$, since by construction $Z, \bigcup \psi \subseteq Z$.

We will now show that Z is an open and maximal branch.

By conclusion 4.54, set Z is a branch. By construction of set Z, Z is an open branch, that is none of the subsets of Z is t-inconsistent, by virtue of definition of set $\overline{X}$.

Let us now check if Z is a maximal branch. Making use of the definition of maximal branch 4.40, assume that there exists a tableau rule $R \in \mathbf{R_{S5}}$ and n-tuple $\langle X_1, \ldots, X_n \rangle \in R$, for some $n \in \mathbb{N}$, such that $X_1 = Z$. By definition of tableau rules 4.23, there exists n-tuple $\langle X'_1 \ldots, X'_n \rangle \in R$ such that for any $1 < j \leq n$, $X_j \setminus X_1 = X'_j \setminus X'_1$ and $X^i \subseteq X'_1 \subseteq Z$. Since Z is a closure under tableau rules of set X^i, so $X'_j \subseteq Z$, for certain $1 < j \leq n$, by definition 4.53. Therefore, also $X_j \subseteq Z$, since $X_j = X_1 \cup X'_j$. But then $X_1 \nsubseteq X_j$, which by definition of tableau rules 4.23 is out of the question. Consequently, there exists no tableau rule and n-tuple $\langle X_1, \ldots, X_n \rangle \in R$ such that $X_1 = Z$, for some $n \in \mathbb{N}$. Therefore, Z is a maximal branch, by definition of maximal branch 4.40 □

Let us define the concept of model generated by branch.

Definition 4.56 (Model generated by branch)**.** Let ϕ be a branch. Let X be a non-empty subset of set of formulas $\mathsf{For_{S5}}$ and $\{\langle A, k \rangle : A \in X\} \subseteq \bigcup \phi$, for some $k \in \mathbb{N}$. We define function $AT(\phi) \subseteq \mathsf{Te_{S5}}$ as follows, $x \in AT(\phi)$ iff one of the below conditions is met:

- $x \in \bigcup \phi \cap \{irj : i, j \in \mathbb{N}\}$
- $x \in \bigcup \phi \cap (\mathsf{Var} \times \mathbb{N})$.

We shall state that model $\mathfrak{M}_{\mathbf{S5}} = \langle W, R, V, w \rangle$ is *generated by branch* ϕ iff:

- $W = \{i : i \in *(AT(\phi))\} \cup \{k\}$
- for any $i, j \in \mathbb{N}$, $\langle i, j \rangle \in R$ iff $irj \in AT(\phi)$
- $V(x, i) = 1$ iff $\langle x, i \rangle \in AT(\phi)$
- $w = k$.

Let model $\mathfrak{M}_{\mathbf{S5}}$ be generated by ϕ. Then, we shall state that ϕ *generates the model.*

From the definition of generated model, another conclusion results.

Corollary 4.57. *Let ϕ be such an open branch that for certain non-empty set of formulas X and some index $i \in \mathbb{N}$, $\{\langle A, i \rangle : A \in X\} \subseteq \bigcup \phi$. Then, ϕ generates model $\mathfrak{M}_{\mathbf{S5}} = \langle W, R, V, i \rangle$.*

Proof. By definition of open branch 4.30, definition of model 4.5 and definition of model generated by branch 4.56. □

Lemma 4.58 (On generation of model)**.** *Let ϕ be an open and maximal branch. Then, for any non-empty $X \subseteq \mathsf{For}_{\mathbf{S5}}$ and any index i such that $\{\langle A,i\rangle : A \in X\} \subseteq \bigcup\phi$ there exists model $\mathfrak{M}_{\mathbf{S5}}$ such that for any formula A, if $A \in X$, then $\mathfrak{M}_{\mathbf{S5}} \vDash A$.*

Proof. Take any maximal and open branch ϕ such that for certain set $X \subseteq \mathsf{For}_{\mathbf{S5}}$ and certain index i, $\{\langle A,i\rangle : A \in X\} \subseteq \bigcup\phi$. Since ϕ is open, then according to conclusion 4.57, there exists, generated by ϕ, model $\mathfrak{M}_{\mathbf{S5}} = \langle W, R, V, w\rangle$, specified in accordance with definition 4.56, where $w = i$.

We will carry out an inductive proof due to the construction of formulas, showing that for any formula E and any $k \in \mathbb{N}$, if $\langle E,k\rangle \in \bigcup\phi$, then $\mathfrak{M}_{\mathbf{S5}} = \langle W, R, V, k\rangle \vDash E$.

Initial step. Take any $x \in \mathsf{Var}$ and some index $j \in \mathbb{N}$.

If $\langle x,j\rangle \in \bigcup\phi$, then — according to the definition of generated model $\mathfrak{M}_{\mathbf{S5}}$ — $V(x,j) = 1$, thus by definition of truth in model 4.7, $\langle W, R, V, j\rangle \vDash x$.

If $\langle \neg x,j\rangle \in \bigcup\phi$, then since branch ϕ is open, $\langle x,j\rangle \notin \bigcup\phi$, and — according to the definition of generated model $\mathfrak{M}_{\mathbf{S5}}$ — $V(x,j) = 0$, thus by definition of truth in model 4.7, $\langle W, R, V, j\rangle \vDash \neg x$.

Induction step. (†) Take any formula $E \in \mathsf{For}_{\mathbf{S5}}$ and indices $j, k \in \mathbb{N}$ and assume that for each tableau expression $\langle D, n\rangle$, where $D \in \mathsf{For}_{\mathbf{S5}}$ and $n \in \mathbb{N}$ that belongs to set $\bigcup\phi$ as a result of application of some tableau rule to set $\{\langle E,j\rangle\} \subseteq \bigcup\phi$ or set $\{\langle E,j\rangle, jrk\} \subseteq \bigcup\phi$, it is the case that $\langle W, R, V, n\rangle \vDash D$.

Making use of the inductive assumption, let us consider all cases of construction of formula E. Take some index $j \in \mathbb{N}$.

1. Let $E = (B \wedge C)$ and $\langle (B \wedge C),j\rangle \in \bigcup\phi$. Then, since ϕ is a maximal branch, due to rule $R_{\wedge}$, both $\langle B,j\rangle$ and $\langle C,j\rangle$ belong to $\bigcup\phi$. From assumption (†) and definition of truth in model 4.7, we get that $\langle W, R, V, j\rangle \vDash (B \wedge C)$.
2. Let $E = (B \vee C)$ and $\langle (B \vee C),j\rangle \in \bigcup\phi$. Then, since ϕ is a maximal branch, due to rule $R_{\vee}$, $\langle B,j\rangle$ or $\langle C,j\rangle$ belongs to $\bigcup\phi$. From assumption (†) and definition of truth in model 4.7, we get that $\langle W, R, V, j\rangle \vDash (B \vee C)$.
3. Let $E = (B \rightarrow C)$ and $\langle (B \rightarrow C),j\rangle \in \bigcup\phi$. Then, since ϕ is a maximal branch, due to rule $R_{\rightarrow}$, $\langle \neg B,j\rangle$ or $\langle C,j\rangle$ belongs to $\bigcup\phi$. From assumption (†) and definition of truth in model 4.7, we get that $\langle W, R, V, j\rangle \vDash (B \rightarrow C)$.
4. Let $E = (B \leftrightarrow C)$ and $\langle (B \leftrightarrow C),j\rangle \in \bigcup\phi$. Then, since ϕ is a maximal branch, due to rule $R_{\leftrightarrow}$, $\langle B,j\rangle$, $\langle C,j\rangle$ belong to $\bigcup\phi$ or $\langle \neg B,j\rangle$, $\langle \neg C,j\rangle$ belong to $\bigcup\phi$. From assumption (†) and definition of truth in model 4.7, we get that $\langle W, R, V, j\rangle \vDash (B \leftrightarrow C)$.
5. Let $E = \neg\neg B$ and $\langle \neg\neg B,j\rangle \in \bigcup\phi$. Then, since ϕ is a maximal branch, due to rule $R_{\neg\neg}$, $\langle B,j\rangle$ belongs to $\bigcup\phi$. From assumption (†) and definition of truth in model 4.7, we get that $\langle W, R, V, j\rangle \vDash \neg\neg B$.

6. Let $E = \neg(B \wedge C)$ and $\langle \neg(B \wedge C), j\rangle \in \bigcup\phi$. Then, since ϕ is a maximal branch, due to rule $R_{\neg\wedge}$, $\langle \neg B, j\rangle$ or $\langle \neg C, j\rangle$ belongs to $\bigcup\phi$. From assumption (†) and definition of truth in model 4.7, we get that $\langle W, R, V, j\rangle \vDash \neg(B \wedge C)$.
7. Let $E = \neg(B \vee C)$ and $\langle \neg(B \vee C), j\rangle \in \bigcup\phi$. Then, since ϕ is a maximal branch, due to rule $R_{\neg\vee}$, $\langle \neg B, j\rangle$ and $\langle \neg C, j\rangle$ belong to $\bigcup\phi$. From assumption (†) and definition of truth in model 4.7, we get that $\langle W, R, V, j\rangle \vDash \neg(B \vee C)$.
8. Let $E = \neg(B \rightarrow C)$ and $\langle \neg(B \rightarrow C), j\rangle \in \bigcup\phi$. Then, since ϕ is a maximal branch, due to rule $R_{\neg\rightarrow}$, $\langle B, j\rangle$ and $\langle \neg C, j\rangle$ belong to $\bigcup\phi$. From assumption (†) and definition of truth in model 4.7, we get that $\langle W, R, V, j\rangle \vDash \neg(B \rightarrow C)$.
9. Let $E = \neg(B \leftrightarrow C)$ and $\langle (B \leftrightarrow C), j\rangle \in \bigcup\phi$. Then, since ϕ is a maximal branch, due to rule $R_{\neg\leftrightarrow}$, $\langle \neg B, j\rangle$, $\langle C, j\rangle$ belong to $\bigcup\phi$ or $\langle B, j\rangle$, $\langle \neg C, j\rangle$ belong to $\bigcup\phi$. From assumption (†) and definition of truth in model 4.7, we get that $\langle W, R, V, j\rangle \vDash (B \leftrightarrow C)$.
10. Let $E = \neg\Box B$ and $\langle \neg\Box B, j\rangle \in \bigcup\phi$. Then, since ϕ is a maximal branch, due to rule $R_{\neg\Box}$, $\langle \Diamond\neg B, j\rangle$ belongs to $\bigcup\phi$. From assumption (†) and definition of truth in model 4.7, we get that $\langle W, R, V, j\rangle \vDash \neg\Box B$.
11. Let $E = \neg\Diamond B$ and $\langle \neg\Diamond B, j\rangle \in \bigcup\phi$. Then, since ϕ is a maximal branch, due to rule $R_{\neg\Box}$, $\langle \Box\neg B, j\rangle$ belongs to $\bigcup\phi$. From assumption (†) and definition of truth in model 4.7, we get that $\langle W, R, V, j\rangle \vDash \neg\Diamond B$.
12. Let $E = \Box B$ and $\langle \Box B, j\rangle \in \bigcup\phi$. We have theoretically two cases: either (i) for none $l \in \mathbb{N}$ expression jrl belongs to $\bigcup\phi$, or (ii) there exists such $l \in \mathbb{N}$ that expression jrl belongs to $\bigcup\phi$. However, case (i) does not hold, because by rule R_r, expression jrj belongs to $\bigcup\phi$ at least. In case (ii) we take set $\{l : jrl \in \bigcup\phi\}$ — by assumption, this set is non-empty. Since branch ϕ is maximal, by virtue of rule $R_\Box$ for any $m \in \{l : jrl \in \bigcup\phi\}$ set $\bigcup\phi$ contains expression $\langle B, m\rangle$. Whereas due to construction of model $\mathfrak{M}_{\mathbf{S5}}$ and (†), we know that $\langle W, R, V, m\rangle \vDash B$. Therefore, by definition of truth in model 4.7, we get that $\langle W, R, V, j\rangle \vDash \Box B$.
13. Let $E = \Diamond B$ and $\langle \Diamond B, j\rangle \in \bigcup\phi$. Since branch ϕ is maximal, due to rule $R_\Diamond$, there exists index $l \in \mathbb{N}$ such that expressions $\langle B, l\rangle$ and jrl belong to $\bigcup\phi$. From the construction of model $\mathfrak{M}_{\mathbf{S5}}$ and (†), we know that $l \in W$, $\langle j, l\rangle \in R$ and $\langle W, R, V, l\rangle \vDash B$. Therefore, by definition of truth in model 4.7, we get $\langle W, R, V, j\rangle \vDash \Diamond B$.

Thus, we have proven that for any formula E and any index j, if $\langle E, j\rangle \in \bigcup\phi$, then $\langle W, R, V, j\rangle \vDash E$. Therefore, there exists such model $\mathfrak{M}_{\mathbf{S5}}$ that for any formula A, if $A \in X$, then $\mathfrak{M}_{\mathbf{S5}} \vDash A$ since $\mathfrak{M}_{\mathbf{S5}} = \langle W, R, V, i\rangle$. □

The above concepts and facts allow us to demonstrate a partial relationship between the relation of semantic consequence and the relation of branch consequence in the tableau system for **S5**.

Lemma 4.59. *For any $X \subseteq \mathsf{For}_{\mathbf{S5}}$, $A \in \mathsf{For}_{\mathbf{S5}}$, if $X \models A$, then $X \rhd A$.*

Proof. Take any $X \subseteq \mathsf{For}_{\mathbf{S5}}$ and $A \in \mathsf{For}_{\mathbf{S5}}$. Assume that $X \not\rhd A$. We must show that $X \not\models A$. From the assumption and definition of $\rhd$, 4.44, we know that there exists no such index $i \in \mathbb{N}$ and such finite set $Y \subseteq X$ that each maximal branch beginning with set $\{\langle B, i\rangle : B \in Y \cup \{\neg A\}\}$ is closed. Therefore, for each index $i \in \mathbb{N}$ and each finite set $Y \subseteq X$, there exists a maximal branch beginning with set $\{\langle B, i\rangle : B \in Y \cup \{\neg A\}\}$ which is open. By lemma 4.55, for some index $i \in \mathbb{N}$, there exists such closure Z of set of expressions $X^i = \{\langle B, i\rangle : B \in X \cup \{\neg A\}\}$ under tableau rules that Z is an open and maximal branch. Since $Z = \bigcup Z$ and $\{\langle B, i\rangle : B \in X \cup \{\neg A\}\} \subseteq Z$, from lemma 4.58, we know that there exists model $\mathfrak{M}_{\mathbf{S5}}$ such that $\mathfrak{M}_{\mathbf{S5}} \models X \cup \{\neg A\}$. Hence, by definition of $\models$, 4.9, $X \not\models A$. □

We will now proceed to the determination of relationship between the branch consequence relation and the existence of a closed tableau. However, this will require some introduction of another concepts.

Let us now define the concept of **R**-branch, that is such branch that originated by the application of rules exclusively from set $\mathbf{R} \subseteq \mathbf{R}_{\mathbf{S5}}$, for some **R**.

Definition 4.60 (**R**-branch)**.** Let $\mathbf{R} \subseteq \mathbf{R}_{\mathbf{S5}}$, let $K = \mathbb{N}$ or $K = \{1, 2, \dots, n\}$, where $n \in \mathbb{N}$. Moreover, let X be a set of expressions. **R**-*branch* (or **R**-*branch beginning with* X) will be called any sequence $\phi : K \longrightarrow P(\mathsf{Te}_{\mathbf{S5}})$ that meets the following conditions:

1. $\phi(1) = X$
2. for any $i \in K$, if $i + 1 \in K$, then there exists such rule $R \in \mathbf{R}$ and such n-tuple $\langle Y_1, \dots, Y_n\rangle \in R$ that $\phi(i) = Y_1$ and $\phi(i+1) = Y_k$, for certain $1 < k \leq n$.

Having established set **R**, the resultant branch will be then called **R**-*branch*.

Definition of **R**-branch differs from definition of branch 4.27 in the fact that the applied rules come from a subset of set of tableau rules $\mathbf{R}_{\mathbf{S5}}$. In a special case when $\mathbf{R} = \mathbf{R}_{\mathbf{S5}}$, both definitions would be identical. But, since set **R** does not have to be identical with set $\mathbf{R}_{\mathbf{S5}}$, so we have a conclusion.

Corollary 4.61. *For any $\mathbf{R} \subseteq \mathbf{R}_{\mathbf{S5}}$, each **R**-branch is a branch.*

In a similar manner, we will define another auxiliary concept, namely the concept of quasi-maximal branch.

Definition 4.62 (Quasi-maximal branch)**.** Let $\mathbf{R} \subseteq \mathbf{R}_{\mathbf{S5}}$ and let $\phi : K \longrightarrow P(\mathsf{Te}_{\mathbf{S5}})$ be a branch. We shall state that ϕ is a *quasi-maximal branch* iff it meets one of the below conditions:

1. ϕ is closed
2. for any rule $R \in \mathbf{R}$, any $n \in \mathbb{N}$ and any n-tuple $\langle X_1, \ldots, X_n \rangle \in R$, if $\phi(k) = X_1$, for certain $k \in K$, then for some $j \in K$, there exist $\phi(j)$ and such set of expressions $W \subseteq \mathsf{Te}_{\mathbf{S5}}$ that for some $1 < i \leq n$, W is strongly similar to X_i (according to $\mathbf{R}$) and $W \subseteq \phi(j)$.

Having established set **R**, the resultant quasi-maximal branch will be called **R**-*quasi-maximal branch*.

The provided definition of quasi-maximal branch also resembles the definition of maximal branch 4.40, while in a special case, when $\mathbf{R} = \mathbf{R_{S5}}$, both definitions would be identical. Again, the difference pertains to the reference to the set of rules which is some subset $\mathbf{R} \subseteq \mathbf{R_{S5}}$, so possibly proper subset of tableau rules. Since a maximal branch must be a sequence closed under all rules, so the relationship that occurs between the quasi-maximal branches and maximal branches is one-directional. That relationship is expressed by another conclusion which follows from the definition of maximal branch 4.40 and definition of quasi-maximal branch 4.62.

Corollary 4.63. *Each maximal branch is* ***R****-quasi-maximal branch, for any* $\mathbf{R} \subseteq \mathbf{R_{S5}}$.

The next conclusion is consequential for further considerations. It follows directly from the definition of quasi-maximal branch 4.62. In the proofs of further facts, the content of that conclusion shall be deemed self-evident.

Corollary 4.64. *For any* $\mathbf{R} \subseteq \mathbf{R_{S5}}$*, each* ***R****-quasi-maximal branch is a branch.*

Let us now introduce the definition of addition of branches.

Definition 4.65 (Addition of branches)**.** Let $\phi : \{1, \ldots, n\} \longrightarrow P(\mathsf{Te}_{\mathbf{S5}})$ and $\psi : M \longrightarrow P(\mathsf{Te}_{\mathbf{S5}})$ be branches, for some $n \in \mathbb{N}$ and $M \subseteq \mathbb{N}$, and let $\phi(n) = \psi(1)$. The results of the operation $\phi \oplus \psi$ is function $\varphi : K \longrightarrow P(\mathsf{Te}_{\mathbf{S5}})$ defined as follows:

1. if $M = \mathbb{N}$, then $K = \mathbb{N}$
2. if $|M| \in \mathbb{N}$, then $K = \{1, \ldots, n, n+1, n+2, \ldots, n+|M|-1\}$
3. for each $i \in K$
   a. if $1 \leq i \leq n$, $\varphi(i) = \phi(i)$
   b. if $i > n$, then $\varphi(i) = \psi((i-n)+1)$.

From definition of addition of branches 4.65, definition of tableau rules 4.23 and definition of branches 4.27, another conclusion follows.

Corollary 4.66. *Let* $\phi : \{1, \ldots, n\} \longrightarrow P(\mathsf{Te}_{\mathbf{S5}})$ *and* $\psi : M \longrightarrow P(\mathsf{Te}_{\mathbf{S5}})$ *be branches, for some* $n \in \mathbb{N}$ *and* $M \subseteq \mathbb{N}$*, and let* $\phi(n) = \psi(1)$*. Then* $\phi \oplus \psi$ *is also a branch.*

Now, we have several facts concerning the relationship between the quasi-maximal branches and the finite sets of expressions.

Proposition 4.67. *Let* $\mathbf{R}_{CPL} = \{R_{\wedge}, R_{\vee}, R_{\rightarrow}, R_{\leftrightarrow}, R_{\neg\neg}, R_{\neg\wedge}, R_{\neg\vee}, R_{\neg\rightarrow}, R_{\neg\leftrightarrow}\}$*. Let* $X \subseteq \mathsf{Te}_{\mathbf{S5}}$ *be a finite set of tableau expressions. Then, there exists a* $\mathbf{R}_{CPL}$*-quasi-maximal branch beginning with set X.*

Proof. Take set of rules $\mathbf{R}_{CPL} = \{R_{\wedge}, R_{\vee}, R_{\rightarrow}, R_{\leftrightarrow}, R_{\neg\neg}, R_{\neg\wedge}, R_{\neg\vee}, R_{\neg\rightarrow}, R_{\neg\leftrightarrow}\}$ and set of tableau expressions $X \subseteq \mathsf{Te}_{\mathbf{S5}}$.

If set X is t-inconsistent, then — by definition 4.62 — one-element sequence $\langle X\rangle$ is a $\mathbf{R}_{CPL}$-quasi-maximal branch beginning with set X. Then, assume that X is not t-inconsistent.

Since X is a finite set, then $*(X)$, that is a set of indices that appear in the expressions in set X (definition of function selecting indices 4.15), is also finite.

If $*(X)$ is an empty set, because $X = \varnothing$, then — by definition 4.62 — one-element sequence $\langle X\rangle$ is a $\mathbf{R}_{CPL}$-quasi-maximal branch beginning with set X.

Assume that $*(X)$ is a non-empty set. By *quasi-modal* formula we will mean each such formula A of logic **S5** that A is different from each following formula $\Diamond B$, $\Box B$, $\neg\Diamond B$, $\neg\Box B$, where B is a formula of **S5**. The set of all quasi-modal formulas will be denoted with symbol $\mathsf{For}^{Q}_{\mathbf{S5}}$. We will divide set of propositional letters Var into two disjoint subsets Var_1 and Var_2 so that propositional letter $x \in \mathsf{Var}_1$ iff $x = p_i$, for some $i \in \mathbb{N}$. From definitions of set Var it follows that both set Var_1 and Var_2 are infinite sets, plus their union equals to set Var.

Since set $\mathsf{For}_{\mathbf{S5}} \smallsetminus \mathsf{For}^{Q}_{\mathbf{S5}}$ and set of propositional letters Var_1 are infinite and countable sets, we can determine bijection $\bullet : \mathsf{For}_{\mathbf{S5}} \smallsetminus \mathsf{For}^{Q}_{\mathbf{S5}} \longrightarrow \mathsf{Var}_1$ that assigns exactly one propositional letter to each formula which is not quasi-modal.

Now, for each index $i \in *(X)$, we define set $X^i = \{\langle A, i\rangle : \langle A, i\rangle \in X$ *and A is a quasi-modal formula*$\}$. Set X^i contains all and only those expressions that belong to set of expressions X which constitute an ordered pair: some quasi-modal formula A, i.e., in terms of structure, formula corresponding to some formula of **CPL**, and index i. Since set X is finite, so for any i, set X^i is also finite.

If for any $i \in *(X)$, set X^i is an empty set, then initial set X does not comprise any subset to which we could apply one of rules $\mathbf{R}_{CPL}$. Therefore — by definition 4.62 — one-element sequence $\langle X\rangle$ is a $\mathbf{R}_{CPL}$-quasi-maximal branch beginning with set X.

Assume that it is not the case that for any $i \in *(X)$, set X^i is an empty set. Now, note that set of rules $\mathbf{R}_{CPL}$ includes analogons of tableau rules from the set of tableau rules for **CPL** — from set $\mathbf{R}_{\mathbf{CPL}}$. In other words, rules for **CPL** "split" formulas into subformulas, and at the same time rules $\mathbf{R}_{CPL}$ for the expressions constructed from quasi-modal formulas and index identically "split" formulas preserving the initial index in the new expression.

For rules $\mathbf{R_{CPL}}$, we have fact 2.34 stating that each finite set of formulas of the classical logic, that is non-modal formulas, is the first element of some such branch of a finite length ϕ that there does not exist super-branch $\phi \subset \psi$.

Now, for any formulas A, B and C, we will define substitution $e : \mathsf{For}_{\mathbf{S5}} \longrightarrow \mathsf{For}_{\mathbf{S5}}$ using the following conditions:

1. if $A \in \mathsf{Var}$, then for any $i \in \mathbb{N}$:
   a. if $A = p_i$ and $i = 1$, then $e(A) = q_1$
   b. if $A = p_i$ and $i \neq 1$, then $e(A) = q_j$, where j is the smallest odd number greater that index in $e(p_{i-1})$
   c. if $A = q_i$, then $e(A) = q_j$, where $j = i \cdot 2$
   d. if $A = r_i$, then $e(A) = r_i$
2. if $A = \neg B$, A is a quasi-modal formula and there is no such formula D that $B = \neg D$, then $e(A) = \neg e(B)$
3. in the other cases:
   a. if $A = \neg\neg B$, then $e(A) = \neg\neg e(B)$
   b. if $A = (B \wedge C)$, then $e(A) = (e(B) \wedge e(C))$
   c. if $A = (B \vee C)$, then $e(A) = (e(B) \vee e(C))$
   d. if $A = (B \rightarrow C)$, then $e(A) = (e(B) \rightarrow e(C))$
   e. if $A = (B \leftrightarrow C)$, then $e(A) = (e(B) \leftrightarrow e(C))$
   f. if $A = \Diamond B$, then $e(A) = \bullet(\Diamond B)$
   g. if $A = \Box B$, then $e(A) = \bullet(\Box B)$
   h. if $A = \neg \Diamond B$, then $e(A) = \bullet(\neg \Diamond B)$
   i. if $A = \neg \Box B$, then $e(A) = \bullet(\neg \Box B)$.

Note that for any formula A, its images under function e, i.e. $e(A)$ is a formula of **CPL**, that is $e(A) \in \mathsf{For}_{\mathbf{CPL}}$. Let us define function $e' : \mathsf{For}_{\mathbf{S5}} \longrightarrow \mathsf{For}_{\mathbf{CPL}}$ with condition: for any formula $A \in \mathsf{For}_{\mathbf{S5}}$, $e'(A) = e(A)$.

Due to the fact that function $\bullet$ is a bijection and that function e is injective, function e' is also a bijection. Hence, there exists inverse function e'^{-1} such that for any formula $A \in \mathsf{For}_{\mathbf{S5}}$, $e'^{-1}(e'(A)) = A$.

For any $i \in \ast(X)$, we now define set $\overline{X^i} = \{e'(A) : \langle A, i\rangle \in X^i\}$.

Obviously, $\overline{X^i}$ is, by virtue of the construction of set X^i, a finite subset of formulas **CPL**. So, from the mentioned fact 2.34, it follows that there exists such a finite branch of length n beginning with set $\overline{X^i}$ that it cannot be anymore extended using the tableau rules for tableau system for **CPL**:

(1) $\overline{X^i_1}, \overline{X^i_2}, \ldots, \overline{X^i_n}$.

Hence, there exists such finite $\mathbf{R}_{CPL}$-branch of length n, beginning with set X^i:

(2) $X_1^i, X_2^i, \ldots, X_n^i$

where for any $1 \leq j \leq n$, $X_j^i = \{\langle A, i\rangle : A \in e'^{-1}(\overline{X_j^i})\}$, that it cannot be anymore extended using the tableau rules from set $\mathbf{R}_{CPL}$. If branch (2) was extendible by some of rules from set $\mathbf{R}_{CPL}$, also branch (1) would be extendible by an equivalent of that rule from set $\mathbf{R_{CPL}}$. But this would contradict fact 2.34.

Note that branch:

$$X_1^i, X_2^i, \ldots, X_n^i$$

is 4.62 quasi-maximal as it was created using rules that belonged to set of rules $\mathbf{R}_{CPL}$ and no rule from set of rules $\mathbf{R}_{CPL}$ can be applied to set X_n^i.

Now, take initial set of expressions X and branch $X_1^i, X_2^i, \ldots, X_n^i$, and for any index $i \in *(X)$ define branch:

$$(\dagger)\ Y_1^i = X_1^i \cup X,\ Y_2^i = X_2^i \cup X, \ldots, Y_n^i = Y_n^i \cup X.$$

Set $Y_1^i = X_1^i \cup X$, by definition of set X_1^i, is equal to set X. So, the defined branch begins with set X.

If for some index $i \in *(X)$, a branch defined with scheme $(\dagger)$ ends with a t-inconsistent set, then according to definition of quasi-maximality 4.62, that branch is quasi-maximal, while since Y_1^i is equal to set X, so there exists a $\mathbf{R}_{CPL}$-quasi-maximal branch beginning with set X.

Let us assume, however, that for no index $i \in *(X)$, a branch defined with scheme $(\dagger)$ ends with a t-inconsistent set.

As we know, set of indices $i \in *(X)$ is finite — assume it contains m indices.

If set $i \in *(X)$ only includes one index, say index $j \in *(X)$, then according to definition of quasi-maximality 4.62 there exists a $\mathbf{R}_{CPL}$-quasi-maximal branch beginning with set X, defined with scheme $(\dagger)$:

$$Y_1^j = X_1^j \cup X,\ Y_2^j = X_2^j \cup X, \ldots, Y_n^j = Y_n^j \cup X.$$

Let us assume, however, that number of indices m that belong to set $*(X)$, is greater than one. Let us arrange the indices that belong to set $i \in *(X)$ in sequence $i_1, i_2, \ldots, i_m$. For indices i_1, i_2, we take two branches, as per scheme $(\dagger)$:

(a) $Y_1^{i_1}, Y_2^{i_1}, \ldots, Y_n^{i_1}$

(b) $Y_1^{i_2}, Y_2^{i_2}, \ldots, Y_l^{i_2}$

and define the third branch, summing up each of sets of branches (b) and last element of branch (a), $Y_n^{i_1}$:

(c) $Y_n^{i_1} \cup Y_1^{i_2}, Y_n^{i_1} \cup Y_2^{i_2}, \ldots, Y_n^{i_1} \cup Y_l^{i_2}$.

Since $Y_1^{i_2} = X$, by definition of branch (†), $Y_n^{i_1} \cup Y_1^{i_2} = Y_n^{i_1}$, because $X \subseteq Y_n^{i_1}$ by definition of branch (†). Now, we will make use of fact on addition of branches 4.66 and define a branch by adding together branches (a) and (c):

(a)⊕(c) $Z_1 = Y_1^{i_1}, Z_2 = Y_2^{i_1}, \ldots, Z_n = Y_n^{i_1}, Z_{n+1} = Y_n^{i_1} \cup Y_2^{i_2}, \ldots, Z_{n+l-1} = Y_n^{i_1} \cup Y_l^{i_2}$.

Branch (a)⊕(c) will be called i_2-branch. Now, assume we have defined i_k-branch for $k < m$, of length $o \in \mathbb{N}$:

(d) $Y_1^{i_k}, Y_2^{i_k}, \ldots, Y_o^{i_k}$,

Next, we take a branch for index i_{k+1}, as per scheme (†):

(e) $Y_1^{i_{k+1}}, Y_2^{i_{k+1}}, \ldots, Y_n^{i_{k+1}}$

and define the third branch, summing up each of sets of branches (e) and last element of branch (d), $Y_o^{i_k}$:

(f) $Y_o^{i_k} \cup Y_1^{i_{k+1}}, Y_o^{i_k} \cup Y_2^{i_{k+1}}, \ldots, Y_o^{i_k} \cup Y_n^{i_{k+1}}$.

Since $Y_1^{i_{k+1}} = X$, by definition of branch (†), $Y_o^{i_k} \cup Y_1^{i_{k+1}} = Y_o^{i_k}$, because $X \subseteq Y_o^{i_k}$ by definition of branch (†) and construction of i_k-branch. Again, we make use of fact on addition of branches 4.66 and define a branch by adding branches (d) and (f):

(d)⊕(f) $Z_1 = Y_1^{i_k}, Z_2 = Y_2^{i_k}, \ldots, Z_o = Y_o^{i_k}, Z_{o+1} = Y_o^{i_k} \cup Y_2^{i_{k+1}}, \ldots, Z_{o+n-1} = Y_o^{i_k} \cup Y_n^{i_{k+1}}$.

Branch (d)⊕(f) will be called i_{k+1}-branch. Carrying out the above actions $m-1$ times, we get i_m-branch, of length n, for some $n \in \mathbb{N}$:

(g) $Y_1^{i_m}, Y_2^{i_m}, \ldots, Y_n^{i_m}$.

We claim that branch (g) is — according to definition of quasi-maximality 4.62 — quasi-maximal, and since $Y_1^{i_m}$ is equal to set X, there exists a $\mathbf{R}_{CPL}$-quasi-maximal branch beginning with set X.

Assume that branch (g) is not closed. Take any rule $R \in \mathbf{R}_{CPL}$ and any such ordered l-tuple $\langle X_1,\dots,X_l\rangle \in R$ that for some $1 \leq j \leq n$, $Y_j^{i_m} = X_1$. Therefore, according to definition of tableau rules 4.23 and set of rules $\mathbf{R}_{CPL}$, in set of tableau expressions $\mathsf{Te}_{\mathbf{S5}}$, there exist such expressions $A_1, \dots, A_l$ that $X_2 \setminus X_1 = \{A_2\}, \dots, X_l \setminus X_1 = \{A_l\}$ and $\langle\{A_1\},\dots,\{A_l\}\rangle \in R$. Since $R \in \mathbf{R}_{CPL}$, so expression A_1 is composed of some formula $B \in \mathsf{For}_{\mathbf{S5}}$ and some index $k \in \mathbb{N}$, thus it has a structure of $\langle B,k\rangle$.

We know that there exists $\mathbf{R}_{CPL}$-quasi-maximal branch:

(h) $Z_1 = \{\langle A,k\rangle : \langle A,k\rangle \in X\}, \dots, Z_{n'}$

for certain $n' \in \mathbb{N}$. Since set Z_1 contains all expressions like $\langle A,k\rangle$ present in set X, so $\langle B,k\rangle$, due to definition of rules $\mathbf{R}_{CPL}$, belongs to some set Z_o, where $1 \leq o \leq n'$ such that Z_o is the first set in branch (h) comprising expression $\langle B,k\rangle$.

From the construction of branch (g), it follows that $Z_o \subseteq Y_j^{i_m}$. And since branch (h) is quasi-maximal, so some of expressions $A_2, \dots, A_l$ belongs to some set $Z_{o+o'}$, where $o+o' \leq n'$. From the construction of branch (g), it follows that $Z_{o+o'} \subseteq Y_{j+o'}^{i_m}$. Due to the fact that $Y_j^{i_m} \subseteq Y_{j+o'}^{i_m}$, set $Y_j^{i_m} \cup \{A_{l'}\} \subseteq Y_{j+o'}^{i_m}$, where $2 \leq l' \leq l$. And since by definition of strong similarity of sets of expressions 4.39, set $Y_j^{i_m} \cup \{A_{l'}\}$ is strongly similar to set $X_{l'}$, so from definition of quasi-maximal branch 4.62, it follows that (g) is $\mathbf{R}_{CPL}$-quasi-maximal. □

Now, we expand the above fact into richer set of rules $\mathbf{R} \subseteq \mathbf{R}_{\mathbf{S5}}$.

Proposition 4.68. *Let* $\mathbf{R}_{CPL\neg\Diamond\neg\Box} = \mathbf{R}_{CPL} \cup \{R_{\neg\Diamond}, R_{\neg\Box}\}$. *Let* $X \subseteq \mathsf{Te}_{\mathbf{S5}}$ *be any finite set of tableau expressions. Then, there exists a* $\mathbf{R}_{CPL\neg\Diamond\neg\Box}$*-quasi-maximal branch beginning with set X.*

Proof. Let $\mathbf{R}_{CPL\neg\Diamond\neg\Box} = \mathbf{R}_{CPL} \cup \{R_{\neg\Diamond}, R_{\neg\Box}\}$. Take any and finite set of expressions $X \subseteq \mathsf{Te}_{\mathbf{S5}}$.

From fact 4.67, we know that there exists a $\mathbf{R}_{CPL}$-quasi-maximal branch beginning with set X:

(a) $Y_1, \dots, Y_n$

where:

1. $\mathbf{R}_{CPL} = \{R_\wedge, R_\vee, R_\to, R_\leftrightarrow, R_{\neg\neg}, R_{\neg\wedge}, R_{\neg\vee}, R_{\neg\to}, R_{\neg\leftrightarrow}\}$
2. $n \in \mathbb{N}$
3. $Y_1 = X$.

If branch (a) is closed, then by definition 4.62, there exists $\mathbf{R}_{CPL\neg\Diamond\neg\Box}$-quasi-maximal branch that begins with set X. Assume, however, that (a) is not a closed branch.

The last element of branch (a), set Y_n, contains a finite number of elements, due to the fact that branch (a) is finite and that for each $0 \leq i < n$, set Y_{i+1} is also finite, by definition of rules $\mathbf{R}_{CPL}$.

Since set Y_n is finite, then for some $m \in \mathbb{N}$, it contains at most m of expressions like $\langle \neg \Diamond A, i \rangle$, where $A \in \mathsf{For}_{\mathbf{S5}}$ and $i \in \mathbb{N}$.

Therefore, making use of rule $R_{\neg\Diamond}$, we can define a branch beginning with set Y_n of length at most $m+1$. Take any such branch of a maximal length:

(b) $Y_n^1, \ldots Y_n^o$

where $o \in \mathbb{N}$ and $o \leq m+1$. Branch (b) is $R_{\neg\Diamond}$-quasi-maximal, by definition 4.62.

In view of the fact that the last element of branch (a) is the first element of branch (b), we can add both branches, by virtue of fact 4.66, to get branch (a)⊕(b):

(c) $Y_1, \ldots, Y_{n+o-1}$

of length $n+o-1$.

If branch (b) is closed, then also branch (c) is closed, and — by definition 4.62 — it is $\mathbf{R}_{CPL\neg\Diamond\neg\Box}$-quasi-maximal, and moreover it begins with set X. Assume, however, that (c) is not a closed branch.

The last element of branch (c), set Y_{n+o-1}, features a finite number of elements, due to the fact that branch (c) is finite and that for each $0 \leq i < n+o-1$, set Y_{i+1} is also finite, by definition of rules $\mathbf{R}_{CPL\neg\Diamond}$.

Since set Y_{n+o-1} is finite, then for some $k \in \mathbb{N}$, it contains at most k of expressions like $\langle \neg \Box A, i \rangle$, where $A \in \mathsf{For}_{\mathbf{S5}}$ and $i \in \mathbb{N}$.

Therefore, making use of rule $R_{\neg\Box}$, we can define a branch beginning with set Y_{n+o-1} of length at most $k+1$. Take any such branch of a maximal length:

(d) $Y_{n+o-1}^1, \ldots Y_{n+o-1}^j$

where $j \in \mathbb{N}$ and $j \leq k+1$. Branch (d) is $R_{\neg\Box}$-quasi-maximal, by definition 4.62.

In view of the fact that the last element of branch (c) is the first element of branch (d), we can add both branches, by virtue of fact 4.66, to get branch (c)⊕(d):

(e) $Y_1, \ldots, Y_{n+o+j-2}$

of length $n+o+j-2$.

If branch (d) is closed, then also branch (e) is closed, and — by definition 4.62 — it is $\mathbf{R}_{CPL\neg\Diamond\neg\Box}$-quasi-maximal, and moreover it begins with set X. Assume, however, that (e) is not a closed branch.

Now, take any rule $R \in \mathbf{R}_{CPL\neg\Diamond\neg\Box}$ and any ordered m-tuple $\langle Z_1, \ldots, Z_m \rangle \in R$, for some $m \in \mathbb{N}$. Assume that set Z_1 is an element of branch (e). By the construction of branch (e), the sequence also includes such set U that for some $1 < i \leq m$, $Z_i \subseteq U$ and U is an element of branch (a), (b) or (d), thus by the construction of branch (e), it is also an element of branch (e). And since set Z_i by virtue of definition of strong similarity of sets of expressions 4.39 is strongly similar to Z_i, due to definition of quasi-maximal branch 4.62, branch (e) is $\mathbf{R}_{CPL\neg\Diamond\neg\Box}$-quasi-maximal branch.

Since branch (e) begins with set of expressions X, so there exists a $\mathbf{R}_{CPL\neg\Diamond\neg\Box}$-quasi-maximal branch beginning with set X. □

Again, we expand the above fact into richer set of rules $\mathbf{R} \subseteq \mathbf{R}_{\mathbf{S5}}$.

Proposition 4.69. *Let $\mathbf{R}_r = \mathbf{R}_{CPL\neg\Diamond\neg\Box} \cup \{R_r\}$. Let $X \subseteq \mathsf{Te}_{\mathbf{S5}}$ be any finite set of tableau expressions. Then, there exists a $\mathbf{R}_r$-quasi-maximal branch beginning with set X.*

Proof. Let $\mathbf{R}_r = \mathbf{R}_{CPL\neg\Diamond\neg\Box} \cup \{R_r\}$. Take any and finite set of expressions $X \subseteq \mathsf{Te}_{\mathbf{S5}}$.

From fact 4.68 we know that there exists a $\mathbf{R}_{CPL\neg\Diamond\neg\Box}$-quasi-maximal branch beginning with set X:

(a) $Y_1, \ldots, Y_n$

where:

1. $\mathbf{R}_{CPL\neg\Diamond\neg\Box} = \{R_\wedge,\ R_\vee,\ R_\rightarrow,\ R_\leftrightarrow,\ R_{\neg\neg},\ R_{\neg\wedge},\ R_{\neg\vee},\ R_{\neg\rightarrow},\ R_{\neg\leftrightarrow} R_{\neg\Diamond}, R_{\neg\Box}, R_\Diamond, R_\Box\}$
2. $n \in \mathbb{N}$
3. $Y_1 = X$.

If branch (a) is closed, then by definition 4.62, there exists $\mathbf{R}_r$-quasi-maximal branch that begins with set X. Assume, however, that (a) is not a closed branch.

The last element of branch (a), set Y_n, contains a finite number of elements, due to the fact that branch (a) is finite and that for each $0 \leq i < n$, set Y_{i+1} is also finite, by definition of rules $\mathbf{R}_{CPL\neg\Diamond\neg\Box}$.

Since set Y_n is finite, then for some $m \in \mathbb{N}$, set $*(Y_n)$ contains at most m indices.

Therefore, making use of rule R_r, we can define a branch beginning with set Y_n of length at most of $(m \cdot m) + 1$, because by definition of rule R_r, if indices i, j

belong to given set $*(Z)$, then that rule makes it possible to add set $Z \cup \{irj\}$ in the branch, as long as $irj \notin Z$.

Take any branch maximal in length.

(b) $Y_n^1, \ldots Y_n^o$

where $o \in \mathbb{N}$ and $o \leq (m \cdot m) + 1$. Branch (b) is R_r-quasi-maximal, by definition 4.62.

In view of the fact that the last element of branch (a) is the first element of branch (b), we can add both branches, by virtue of fact 4.66, to get branch (a)⊕(b):

(c) $Y_1, \ldots, Y_{n+o-1}$

of length $n + o - 1$.

If branch (b) is closed, then also branch (c) is closed, and — by definition 4.62 — it is $\mathbf{R}_r$-quasi-maximal, and moreover it begins with set X. Assume, however, that (b) is not a closed branch.

Now, take any rule $R \in \mathbf{R}_r$ and any ordered m-tuple $\langle Z_1, \ldots, Z_m \rangle \in R$, for some $m \in \mathbb{N}$. Assume that set Z_1 is an element of branch (c). By the construction of branch (c), that sequence also includes such set U that for some $1 < i \leq m$, $Z_i \subseteq U$ and U is an element of branch (a) or (b), thus by the construction of branch (c), it is also an element of branch (c). And since set Z_i by virtue of by definition of strong similarity of sets of expressions 4.39 is strongly similar to Z_i, due to definition of quasi-maximal branch 4.62, branch (c) is $\mathbf{R}_r$-quasi-maximal branch.

Since branch (c) begins with set of expressions X, so there exists a $\mathbf{R}_r$-quasi-maximal branch beginning with set X. □

In order to extend the latter fact onto further rules from set $\mathbf{R}_{\mathbf{S5}}$, we need additional definitions. Prior to expressing the definition of cycle of rules $R_\Box$, we will first explain the idea of cyclical application of that rule.

Let $X \subseteq \mathsf{Te}_{\mathbf{S5}}$ be a finite set of tableau expressions. Let the following branch (a) $X_1, \ldots, X_n$, where $n \in \mathbb{N}$ and $X_1 = X$, be $\mathbf{R}_r$-quasi-maximal. We specify two sets of expressions:

$$X' = \{irj : i, j \in \mathbb{N}, irj \in X_1\}$$

$$X'' = \{\langle \Box A, i \rangle : \Box A \in \mathsf{For}_{\mathbf{S5}}, i \in \mathbb{N}, \text{and } \langle \Box A, i \rangle \in X_1\}.$$

Union $X' \cup X''$ is a finite set because set X_1 is finite by assumption. Moreover, union $X' \cup X'' \subseteq X_n$ because $X_1 \subseteq X_n$.

Set $X' \cup X''$ contains a finite number of such two-element subsets $\{\langle \Box A, i\rangle, irj\}$ that if expression $\langle A, j\rangle$ does not belong to t-consistent set of tableau expressions Y, but set Y contains set $\{\langle \Box A, i\rangle, irj\}$, then pair $\langle Y, Y \cup \{\langle A, j\rangle\}\rangle \in R_\Box$. Thus, in particular, if expression $\langle A, j\rangle$ does not belong to set X_n and set X_n is t-consistent, then pair $\langle X_n, X_n \cup \{\langle A, j\rangle\}\rangle \in R_\Box$. Assume that the number of subsets $\{\langle \Box A, i\rangle, irj\} \subseteq X' \cup X''$ is l, for some $l \geq 0$.

Now, we extend branch (a) by rule $R_\Box$, taking account of all l sets $\{\langle \Box A, i\rangle, irj\} \subseteq X' \cup X''$, in an arbitrary order. Thereby, we get a branch of length at most $m \leq n + l$:

(b) $X_1, \ldots, X_n, X_{n+1} = X_n \cup \{\langle A_{n+1}, i_{n+1}\rangle\}, \ldots, X_m = X_{m-1} \cup \{\langle A_m, i_m\rangle\}$

where for any $n < j \leq m$, if set X_j belongs to branch (b), then set X_{j-1} is t-consistent and expression $\langle A_j, i_j\rangle$ does not belong to set X_{j-1}.

If set $\{\langle \Box A, k\rangle, kro\} \subseteq X' \cup X''$ does not exist, for some indices k, o, such that pair $\langle X_m, X_{m+1} \cup \{A, o\}\rangle \in R_\Box$, then branch (b) cannot be anymore extended using rule $R_\Box$ by applying it to some pair from set $\{\{x, y\} : x \in X', y \in X''\} \subseteq X_1$. This does not mean, of course, that there no new pairs appeared in the branch, in the subsequent elements of the branch to which we could apply rule $R_\Box$. Nevertheless, we have exhausted all the initial possibilities, closing a certain stage the result of which will be called the cycle of rule $R_\Box$. Let us now proceed to a formal definition.

Definition 4.70 (Cycle of rule $R_\Box$)**.** Let X be a finite set of expressions. Let branch ϕ: $X_1, \ldots, X_n$ be such branch that $X_1 = X$ and $n \in \mathbb{N}$. Branch ϕ will be called a *cycle of rule* $R_\Box$ iff the below conditions are met:

1. for certain $m \leq n$, branch $X_1, \ldots, X_m$ is $\mathbf{R}_r$-quasi-maximal
2. for each $m < l \leq n$ there exist such indices $o, k \in \mathbb{N}$ and formula A that $X_l = X_{l-1} \cup \{\langle A, o\rangle\}$ and $\{\langle \Box A, k\rangle, kro\} \subseteq X_1$
3. there is no set $\{\langle \Box A, k\rangle, kro\} \subseteq X_1$, for some indices k, o and formula A, such that pair $\langle X_n \cup \{\langle \Box A, k\rangle, kro\}, X_n \cup \{\langle A, o\rangle\}\rangle \in R_\Box$.

Another fact follows from fact 4.69 and definition 4.70.

Proposition 4.71. *Let $X \subseteq \mathsf{Te}_{\mathbf{S5}}$ be a finite set of tableau expressions. Then, there exists cycle of rule $R_\Box$.*

Now, we expand the concept of cycle onto rule $R_\Diamond$, which will make our considerations to cover all the rules from set of tableau rules $\mathbf{R_{S5}}$. Prior to that, however, let us look into the issue in a similar way as the expansion onto rule $R_\Box$.

Let $X \subseteq \mathsf{Te}_{\mathbf{S5}}$ be a finite set of tableau expressions. Let branch (a) $X_1, \ldots, X_n$, where $n \in \mathbb{N}$ and $X_1 = X$, be a cycle of rule $R_\Box$.

We specify set $X' = \{\langle \Diamond A, i\rangle : A \in \mathsf{For}_{\mathbf{S5}}, i \in \mathbb{N} \text{ and } \langle \Diamond A, i\rangle \in X_1\}$.

Set X' is a finite set because set X_1 is finite by assumption. Moreover, $X' \subseteq X_n$ because $X_1 \subseteq X_n$.

Set X' contains a finite number of such expressions $\langle \Diamond A, k\rangle$ that if subset $\{\langle A, j\rangle, krj\}$ is not contained in t-consistent set of tableau expressions Y and $j \notin *(Y)$, but $\langle \Diamond A, k\rangle \in Y$, then pair $\langle Y, Y \cup \{\langle A, j\rangle, krj\}\rangle \in R_{\Diamond}$. Thus, in particular, if for any index j subset $\{\langle A, j\rangle, krj\}$ is not contained in set X_n, X_n is t-consistent and $j \notin *(X_n)$, then pair $\langle X_n, X_n \cup \{\langle A, j\rangle, krj\}\rangle \in R_{\Diamond}$. Assume that the number of such expressions in set X' is $l \geq 0$.

Now, we extend branch (a) by rule $R_{\Diamond}$, taking account of all l expressions $\langle \Diamond A, k\rangle \in X'$, in an arbitrary order. Thereby, we get a branch of length at most $m \leq n + l$:

(b) $X_1, \ldots, X_n, X_{n+1} = X_n \cup \{\langle A_{n+1}, i_{n+1}\rangle, k_n r i_{n+1}\}, \ldots, X_m = X_{m-1} \cup \{\langle A_m, i_m\rangle, k_{m-1} r i_m\}$

where for any $n < j \leq m$, if set X_j belongs to branch (b), then X_{j-1} is t-consistent, $\{\langle \Diamond A_j, k_{j-1}\rangle\} \in X_{j-1}$, set $\{\langle A_j, o\rangle, k_{j-1} r o\}$ is not contained in set X_{j-1} for any $o \in \mathbb{N}$, and $o \notin *(X_{j-1})$.

If there is no tableau expression $\langle \Diamond A, k\rangle \in X'$, for some index k, such that pair $\langle X_m, X_{m+1} \cup \{\langle A, o\rangle, kro\}\rangle \in R_{\Diamond}$, for some index o, then branch (b) cannot be anymore extended using rule $R_{\Diamond}$ by applying it to the expressions from set X_1. Again, this does not mean, of course, that there no new pairs appeared in the branch, in the subsequent elements of the branch to which we could apply rule $R_{\Diamond}$. However, we have exhausted all the initial possibilities, closing a certain stage the result of which will be called the cycle, precisely. Let us now proceed to a formal definition of cycle.

Definition 4.72 (Cycle)**.** Let X be a finite set of expressions. Let branch ϕ: $X_1, \ldots, X_n$ be such branch that $X_1 = X$ and $n \in \mathbb{N}$. Branch ϕ will be called a *cycle* iff the below conditions are met:

1. for certain $m \leq n$, branch $X_1, \ldots, X_m$ is a cycle of rule $R_{\Box}$
2. for each $m < l \leq n$ there exist such indices $o, k \in \mathbb{N}$ and formula A that $X_l = X_{l-1} \cup \{\langle A, o\rangle, kro\}$ and $\{\langle \Diamond A, k\rangle\} \subseteq X_1$
3. there is no set $\{\langle \Diamond A, k\rangle\} \subseteq X_1$, for some index k and formula A, such that for some index o pair $\langle X_n \cup \{\langle \Diamond A, k\rangle\}, X_n \cup \{\langle \Diamond A, k\rangle, kro, \langle A, o\rangle\}\rangle \in R_{\Diamond}$.

Another fact follows from fact 4.71 and definition 4.72.

Proposition 4.73. *Let $X \subseteq \mathsf{Te}_{\mathbf{S5}}$ be a finite set of tableau expressions. Then, there exists a cycle.*

Now, we will proceed to the principal fact from among those concerning quasi-maximal branches. We will show that for a finite set of expressions, there exists **R**-quasi-maximal branch that begins with that set, for $\mathbf{R} = \mathbf{R}_{\mathbf{S5}}$, thus by definition of maximal branch 4.40 and definition of quasi-maximal branch 4.62, a maximal branch.

Proposition 4.74. *Let $X \subseteq \mathsf{Te}_{\mathbf{S5}}$ be a finite set of tableau expressions. Then, there exists a maximal branch beginning with set X.*

Proof. Take any finite set of expressions $X \subseteq \mathsf{Te}_{\mathbf{S5}}$. From fact 4.73, it follows that there exists cycle:

(1) $X_1^1, \ldots, X_o^1$

where:

1. $o \in \mathbb{N}$
2. $X_1^1 = X$
3. all elements of the sequence have originated by the rules applicable for set X_1^1 from set of tableau rules $\mathbf{R}_{\mathbf{S5}}$.

(†) For any $n > 1$, we now define the following cycle:

(n) $X_1^n, \ldots, X_m^n$

where:

1. $m \in \mathbb{N}$
2. $X_1^n = X_k^{n-1}$ and X_k^{n-1} is the last element of cycle $(n-1)$ which is k long, for certain $k \in \mathbb{N}$
3. all elements of the sequence have originated through the rules applicable for set X_1^n from set of tableau rules $\mathbf{R}_{\mathbf{S5}}$.

There may exist an infinite number of cycles such that the last element of cycle (n) is the first element of cycle $(n+1)$, and there exists at least one cycle like (1), that is such that the first element of that cycle is set of tableau expressions X. From the set of all cycles, we select the minimal set of cycles **C** such that:

1. precisely one cycle like (1) belongs to **C**
2. if cycle (n) belongs to **C** and set Z is the last element of cycle (n), then set **C** contains cycle $(n+1)$ with set Z as the first element.

So, in set **C** for each (n) there exists precisely one cycle.

Now, assume that in set **C** there exists one-element cycle. Let it be cycle (k). Therefore by definition of cycle 4.72 and definition (†), for each $i \in \mathbb{N}$, each cycle $(k+i)$ has one element, what is more, cycle (k) is identical to cycle $(k+i)$.

In such case, on cycles from (1) to (k), making use $k-1$ times of conclusion concerning addition of branches 4.66, we define branch:

$$(\ddagger)\ (\ldots(\underbrace{X_1^1,\ldots,X_n^1}_{(1)} \oplus \underbrace{X_1^2,\ldots,X_m^2}_{(2)}) \oplus \ldots) \oplus \underbrace{X_1^k}_{(k)}$$

where $n,m,\ldots,o$ are the lengths of individual cycles.

Assume that branch $(\ddagger)$ is not closed. We claim that branch $(\ddagger)$ is maximal. Because, take any rule $R \in \mathbf{R_{S5}}$ and any j-tuple $\langle Z_1,\ldots,Z_j\rangle \in R$ such that set Z_1 is an element of branch $(\ddagger)$. If there is no such element W of branch $(\ddagger)$ that some set $U \subseteq W$ is strongly similar, within the meaning of definition of strong similarity 4.39, to set of expressions Z_i, for some $1 < i \leq j$, then from definition of tableau rules 4.23, it follows that rule R contains j-tuple $\langle Z_1 \cup X_1^k,\ldots,Z_j' \cup X_1^k\rangle$, where for any $1 < i \leq j$, $Z_i' \cup X_1^k$ is a set of expressions similar to set $Z_i \cup X_1^k$. What follows further, either (k) is not a one-element cycle or it is not a cycle. This, however, contradicts the construction of set of cycles $\mathbf{C}$.

Now, assume that in set $\mathbf{C}$ there does not exist one-element cycle. Then, each cycle contained in $\mathbf{C}$ has at least two elements. By definition of set $\mathbf{C}$ — as it is the minimal set of cycles — for each cycle (k), cycle $(k+1)$ differs from it since at least its second element is a superset of the last element of cycle (k).

Now, we will arrange all cycles from set $\mathbf{C}$ as per their numbers in the following sequence of sequences:

$$(\ddagger\ddagger)\ \underbrace{X_1^1,\ldots,X_n^1}_{(1)}, \underbrace{X_1^2,\ldots,X_m^2}_{(2)}, \ \ldots, \underbrace{X_1^k,\ldots,X_o^k}_{(k)}, \ldots$$

where $n,m,\ldots,o,\ldots$ are the lengths of individual cycles.

Next, for any cycle (i), where $1 < i$, we remove set X_1^i — this element is identical to the last element of cycle $(i-1)$ — to get sequence:

$$(\ddagger\ddagger\ddagger)\ \underbrace{X_1^1,\ldots,X_n^1}_{(1)}, \underbrace{X_2^2,\ldots,X_m^2}_{(2)}, \ \ldots, \underbrace{X_2^k,\ldots,X_o^k}_{(k)}, \ldots$$

where $n,m,\ldots,o,\ldots$ are the lengths of individual cycles.

That sequence is an infinite branch. It can also be defined as follows. Take function $\phi : \mathbb{N} \longrightarrow P(\mathsf{Te_{S5}})$ specified by the below conditions:

1. $\phi(1) = X_1^1$
2. for any $i,j,o \in \mathbb{N}$, if $\phi(i) = X_j^o$, then:

a. $\phi(i+1) = X^o_{j+1}$, if element X^o_{j+1} belongs to $(\ddagger\ddagger\ddagger)$
b. $\phi(i+1) = X^{o+1}_2$, otherwise.

By definition of sequence $(\ddagger\ddagger\ddagger)$ and definition of branch 4.27, sequence ϕ is a branch since for any $i \in \mathbb{N}$, there exists such rule $R \in \mathbf{R_{S5}}$ and there exists such l-tuple $\langle Y_1, \ldots, Y_l \rangle$ that $\phi(i) = X_1$ i $\phi(i+1) = Y_{l_1}$, where $1 < l_1 \leq l$.

As branch ϕ is infinite, so it is not closed, by fact 4.31.

We claim that branch ϕ is maximal. Because, take any rule $R \in \mathbf{R_{S5}}$ and any n-tuple $\langle Z_1, \ldots, Z_n \rangle \in R$ such that set Z_1 is an element of branch ϕ. So, by definition of branch ϕ, set Z_1 is an element of some cycle (k).

If there is no such element W of cycle (k) that for some set $U \subseteq W$, set U is strongly similar, within the meaning of definition of strong similarity 4.39, to set of expressions Z_i, for some $1 < i \leq n$, then from definition of cycles $(\dagger)$ and definition of tableau rules 4.23 it follows that rule R contains n-tuple $\langle Z_1 \cup X^{k+1}_2, \ldots, Z'_n \cup X^{k+1}_2 \rangle$, where for any $1 < i \leq n$, $Z'_i \cup X^{k+1}_2$ is a set of expressions similar to set $Z_i \cup X^{k+1}_2$. Thus, by definition of cycle 4.72, it follows that cycle $(k+1)$ contains set of expressions W such that for some set $U \subseteq W$, set U is strongly similar, within the meaning of definition of strong similarity 4.39, to set of expressions Z_i, for some $1 < i \leq n$.

From the arbitrariness of rule R and set Z_1, it follows that branch ϕ is a maximal branch. □

The above concepts and facts let us show a partial relationship between the branch consequence relation and the existence of a closed tableau in the tableau system for **S5**.

Lemma 4.75. *For any $X \subseteq \mathsf{For}_{\mathbf{S5}}$, $A \in \mathsf{For}_{\mathbf{S5}}$, if $X \rhd A$, then there exists finite subset $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi \rangle$.*

Proof. Take any $X \subseteq \mathsf{For}_{\mathbf{S5}}$ and $A \in \mathsf{For}_{\mathbf{S5}}$. Assume that $X \rhd A$. Therefore, by definition of $\rhd$, there exists such finite set $Y \subseteq X$ and such index $i \in \mathbb{N}$ that each maximal branch beginning with set $\{\langle B, i \rangle : B \in Y \cup \{\neg A\}\}$ is closed. Thus, by fact 4.74 and by definition of complete tableau 4.50, there exists such non-empty subset Φ of set of branches beginning with set $\{\langle B, i \rangle : B \in Y \cup \{\neg A\}\}$ that $\langle Y, A, \Phi \rangle$ is a closed tableau. □

We will now proceed to the description of dependencies between the existence of a closed tableau and the semantic consequence in the **S5**. However, this will require a determination of several more fundamental facts.

Lemma 4.76. *Let $X \subseteq \mathsf{For}_{\mathbf{S5}}$ be a finite set of formulas, $A \in \mathsf{For}_{\mathbf{S5}}$ and $i \in \mathbb{N}$. If there exists a maximal and open branch beginning with set $\{\langle B,i\rangle : B \in X \cup \{\neg A\}\}$, then each complete tableau $\langle X,A,\Phi\rangle$ is open.*

Proof. Take finite set $X \subseteq \mathsf{For}_{\mathbf{S5}}$, any formula $\neg A \in \mathsf{For}_{\mathbf{S5}}$ and index $i \in \mathbb{N}$ such that there exists a maximal and open branch beginning with set $X^i = \{\langle B,i\rangle : B \in X \cup \{\neg A\}\}$. We will denote that branch with letter ϕ.

$(*)$ Since branch ϕ is open, so no element ϕ is a t-inconsistent set, by definition 4.30.

$(**)$ Since branch ϕ is maximal and open, so for any rule $R \in \mathbf{R_{S5}}$, any $n \in \mathbb{N}$ and any element $Y \in \phi$, if $\langle Y, Y_1, \ldots, Y_n\rangle \in R$, then there exists some element $Z \in \phi$ such that some subsist $W \subseteq Z$ is a set strongly similar to set Y_i, for certain $1 \leq i \leq n$, by definition of maximal branch 4.40.

Now, we indirectly assume that there exists complete and closed tableau $\langle X,A,\Psi\rangle$.

Since tableau $\langle X,A,\Psi\rangle$ is complete, so Ψ is such a minimal subset of set of all maximal branches that $\langle X,A,\Psi\rangle$ is a complete tableau, by definition of complete tableau 4.50.

Since tableau $\langle X,A,\Psi\rangle$ is closed, so each branch that belongs to Ψ, is closed, by definition of closed tableau 4.51. For certain $k \in \mathbb{N}$, each of these branches:

- begins with set $X^k = \{\langle B,k\rangle : B \in X \cup \{\neg A\}\}$, by definition of tableau 4.47,
- and its last element is a t-inconsistent set of expressions, by definition of closed tableau 4.51.

We intend to show that there exists some branch χ such that $\chi \notin \Psi$ and $\langle X,A,\Psi \cup \{\chi\}\rangle$ is a tableau, which contradicts the assumption that $\langle X,A,\Psi\rangle$ is a complete tableau if χ is not a redundant variant of any branch which belongs to Φ.

To this end, we will apply the induction through the branch length in order to construct infinite branches beginning with set X^k. The construction method for such branches will be denoted as $(\dagger)$.

Consider the first element of each branch contained in set of branches Ψ. It is set $X_1 = X^k = \{\langle B,k\rangle : B \in X \cup \{\neg A\}\}$. X_1 is a similar set of expressions — within the meaning of definition of similarity 4.16 — to set $X^i = \{\langle B,i\rangle : B \in X \cup \{\neg A\}\}$. Since $X^i \in \phi$ and branch ϕ is open, so X^i and X_1 are t-consistent, by conclusion 4.19.

Nevertheless, due to the fact that Ψ is a set of closed branches and the considered tableau $\langle X,A,\Psi\rangle$ is complete, there must exist a tableau rule $R \in \mathbf{R_{S5}}$ such that $\langle X_1, Z_2, \ldots, Z_l\rangle \in R$, where $1 < l$, and for each $1 < j \leq l$ there exists such branch in set Ψ that Z_j belongs to that branch, by definition of complete tableau 4.50.

Nonetheless, certain set Z_m — for $1 < m \leq l$ — must be t-consistent. As due to definition of tableau rules 4.23, there exists such l-tuple that $\langle Y_1, \ldots, Y_l \rangle \in R$, where Z_m is a similar set — in the sense of definition of similarity 4.16 — to some set $W_m \subseteq Y_m$ and it is t-consistent since $Y_m \subseteq U \in \phi$, for certain $U \subseteq \mathsf{Te}_{\mathbf{S5}}$, by the fact that ϕ is an $(*)$ open and $(**)$ maximal branch. Set Z_m will be denoted as X_2, while element W_m as X_2^*.

Therefore, for number 1 there exist such branches ψ_1, $\psi_2 \in \Psi$ that:

- $X_1 \in \psi_1$
- set X_2 originated by the application of certain rule $R \in \mathbf{R_{S5}}$ to set X_1, ultimately producing a second element of branch $\psi_2 \in \Psi$
- $X_2 \in \psi_2$
- X_2 is a t-consistent set
- $X_1 \subset X_2$
- for some $j \in \mathbb{N}$, set $X_2^* \subseteq X_j \in \phi$, moreover set X_2^* is similar, within the meaning of definition of similarity 4.16 — to set X_2.

Now, assume that for certain $n \in \mathbb{N}$ there exist such branches $\psi_1, \ldots, \psi_n \in \Psi$ that:

- for any $1 < j \leq n$, set X_j originated through the application of certain rule $R \in \mathbf{R_{S5}}$ to set X_{j-1}, ultimately producing j-th element of branch $\psi_j \in \Psi$
- $X_n \in \psi_n$
- X_n is a t-consistent set
- $X_1 \subset X_2 \subset \ldots \subset X_n$
- for some $i \in \mathbb{N}$, set $X_n^* \subseteq X_i \in \phi$, moreover X_n^* is similar, within the meaning of definition of similarity 4.16 — to set X_n.

Nevertheless, due to the fact that Ψ is a set of closed branches, the considered tableau $\langle X, A, \Psi \rangle$ is complete and set X_n is a t-consistent set, there must exist a tableau rule $R \in \mathbf{R_{S5}}$ such that $\langle X_n, Z_2, \ldots, Z_l \rangle \in R$, where $1 < l$, and for each $1 < j \leq l$ there exists such branch in set Ψ that Z_j belongs to that branch, by definition of complete tableau 4.50.

Nonetheless, certain set Z_m — for $1 < m \leq l$ — must be t-consistent. As due to definition of tableau rules 4.23, there exists such l-tuple that $\langle Y_1, \ldots, Y_l \rangle \in R$, where Z_m is a similar set — within the meaning of definition of similarity 4.16 — to some set $W_m \subseteq Y_m$ and it is t-consistent since $Y_m \subseteq U \in \phi$, for certain $U \subseteq \mathsf{Te}_{\mathbf{S5}}$, by virtue of the fact that ϕ is an $(*)$ open and $(**)$ maximal branch. Set Z_m will be denoted as X_{n+1}, while element W_m as X_{n+1}^*.

Thus, fo any $n \in \mathbb{N}$, there exist such branches $\psi_1, \ldots, \psi_n, \psi_{n+1} \in \Psi$ that:

- for any $1 < j \leq n+1$, set X_j originated through the application of certain rule $R \in \mathbf{R_{S5}}$ to set X_{j-1}, ultimately producing j-th element of branch $\psi_j \in \Psi$
- $X_{n+1} \in \psi_{n+1}$
- X_{n+1} is a t-consistent set
- $X_1 \subset X_2 \subset \ldots \subset X_n \subset X_{n+1}$
- for some $i \in \mathbb{N}$, set $X^*_{n+1} \subseteq X_i \in \phi$, moreover X^*_{n+1} is similar, within the meaning of definition of similarity 4.16 — to set X_{n+1}.

Set of all sets that originate this way $X_1 \subset X_2 \subset \ldots \subset X_n \subset X_{n+1} \subset \ldots$ will be denoted as **X**. Set **X** contains at least one branch ψ such that for any $i \in \mathbb{N}$, if $X_i \in \psi$, then there exists set $X_i \in \mathbf{X}$.

Branch ψ can be defined through the specification of such minimal subset of **X**, set $\mathbf{X}'$ that:

1. $X_1 \in \mathbf{X}'$
2. for any $i \in \mathbb{N}$, if $X_i \in \mathbf{X}'$, then exactly one $X_{i+1} \in \mathbf{X}'$.

Branch ψ is infinite, and as a consequence of conclusion 4.31 it is an open branch.

Since set X_1, the first element of branch ψ, is equal to set X^k, and moreover for any element $X_i \in \psi$, where $i > 1$, there exists such rule $R \in \mathbf{R_{S5}}$ and such n-tuple $\langle Y_1, \ldots Y_n \rangle \in R$ that:

- $Y_1 = X_{i-1}$
- $X_i = Y_k$, for certain $1 < k \leq n$
- for each $1 < j \leq n$, if $j \neq k$, then there exists branch $\psi' \in \Psi$ such that for some Z_l, where $1 \leq l$, $Z_l \in \psi'$, $Z_l = Y_1$ and $Z_{l+1} = Y_j$,

so $\langle X, A, \Psi \cup \{\psi\} \rangle$ by virtue of definition of tableau 4.47 is a tableau for pair $\langle X, A \rangle$.

However, branch ψ does not belong to set Ψ because tableau $\langle X, A, \Psi \rangle$, contrary to the assumption, would not be a closed tableau.

Let us now consider the question whether or not set $\Psi \cup \{\psi\}$ is a redundant superset of set Ψ, in the light of definition of redundant variant of branch 4.49. Let us now carry out the following argument.

(††) Assume that branch ψ is a redundant variant of some branch $\psi' \in \Psi$. Thus, for certain minimal $1 \leq i \in \mathbb{N}$ there exist two such rules $R, R' \in \mathbf{R_{S5}}$, ordered couple $\langle U_1, U_2 \rangle \in R$ and ordered triple $\langle W_1, W_2, W_3 \rangle \in R'$, such that:

- $U_1 = W_1$
- $U_1 = X_i \in \psi$ and $U_1 = Y_i \in \psi'$
- $U_2 = W_2$ or $U_2 = W_3$

- (a) if $U_2 = W_2$, then $U_2 = Y_{i+1}$ and $X_{i+1} = W_3$
- (b) if $U_2 = W_3$, then $U_2 = Y_{i+1}$ and $X_{i+1} = W_2$.

No matter which case occurs, (a) or (b), since branch ϕ is open and maximal (assumptions $(*)$ and $(**)$), so also element Y_{i+1} that belongs to ψ' is t-consistent because it is similar to certain set of expressions W included in certain element of branch ϕ.

Therefore, we can construct another infinite and open branch $Z_1, \ldots, Z_{i+1}, \ldots$, making use of construction $(\dagger)$ for which, by virtue of reasoning analogous to $(\dagger\dagger)$, there exists such subsequent element U_{i+2} that $Z_1, \ldots Z_{i+1}, U_{i+2}$ is a t-consistent branch and is not a redundant variant of any sub-branch of any branch from set Ψ.

So, by application of inductive reasoning and steps $(\dagger)$ and $(\dagger\dagger)$ we get an infinite branch — call it χ — and, consequently, open which is not a redundant variant of any branch that belongs to set of branches Ψ and begins with set X_1.

Since Ψ, by assumption, contains closed branches, $\chi \notin \Psi$. Since set X_1, the first element of branch χ, is equal to set X^k, and moreover for any element $X_i \in \chi$, where $i > 1$, there exists such rule $R \in \mathbf{R_{S5}}$ and such n-tuple $\langle Y_1, \ldots Y_n \rangle \in R$ that:

- $Y_1 = X_{i-1}$
- $X_i = Y_k$, for certain $1 < k \leq n$
- for each $1 < j \leq n$, if $j \neq k$, then there exists branch $\psi' \in \Psi$ such that for some Z_l, where $1 \leq l$, $Z_l \in \psi$, $Z_l = Y_1$ and $Z_{l+1} = Y_j$,

so $\langle X, A, \Psi \cup \{\chi\} \rangle$ by virtue of definition of tableau 4.47 is a tableau for pair $\langle X, A \rangle$.

Thus, $\langle X, A, \Psi \rangle$ is not a complete tableau which contradicts the initial assumption. □

In the next fact, we will take up the relationship between the rules from set $\mathbf{R_{S5}}$ and models. That fact states that rules have the following property: for any model, if the model is appropriate for the input set of the rule, then it is also appropriate for at least one output set of that rule.

Proposition 4.77. *Let $X \subseteq \mathsf{Te}_{S5}$, $\mathfrak{M}_{S5}$ be a model and rule $R \in \boldsymbol{R}_{S5}$. If $\langle X, X_1, \ldots, X_n \rangle \in R$, where $1 \leq n \leq 2$, and model $\mathfrak{M}_{S5}$ is appropriate for set X, then $\mathfrak{M}_{S5}$ is appropriate for some set X_i, where $1 \leq i \leq n$.*

Proof. Take any: $X \subseteq \mathsf{Te}_{S5}$, model $\mathfrak{M}_{S5} = \langle W, Q, V, w \rangle$, rule $R \in \mathbf{R_{S5}}$, and $n+1$-tuple $\langle X, X_1, \ldots, X_n \rangle \in R$, for some $1 \leq n \leq 2$.

From definition of model appropriate for the set of expressions 4.20, we know that model $\mathfrak{M}_{S5}$ is appropriate for X iff there exists such function $f : \mathbb{N} \longrightarrow W$ that for any $A \in \mathsf{For}_{S5}$ and $i, j \in \mathbb{N}$:

- if $\langle A, i\rangle \in X$, then $\langle W, Q, V, f(i)\rangle \vDash A$
- if $irj \in X$, then $f(i)Qf(j)$.

Let us consider all possible cases of rule R, and $n+1$-tuples $\langle X, X_1, \ldots, X_n\rangle \in R$, where $1 < n \leq 2$. Take any two formulas $A, B \in \mathsf{For}_{\mathbf{S5}}$, any indices $i, j \in \mathbb{N}$ and any set of expressions $Y \subset X$.

1. Assume that $R = R_{\wedge}$ and $X = Y \cup \{\langle (A \wedge B), i\rangle\}$, $X_1 = X \cup \{\langle A, i\rangle, \langle B, i\rangle\}$. Assume that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X. Since model $\mathfrak{M}_{\mathbf{S5}} = \langle W, Q, V, w\rangle$ is appropriate for set X, there exists such function $f : \mathbb{N} \longrightarrow W$ that $\langle W, Q, V, f(i)\rangle \vDash (A \wedge B)$. Therefore, by definition of truth in model 4.7, $\langle W, Q, V, f(i)\rangle \vDash A$ and $\langle W, Q, V, f(i)\rangle \vDash B$. Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1, it is appropriate for some set X_i, where $1 \leq i \leq n$.
2. Assume that $R = R_{\vee}$ and $X = Y \cup \{\langle (A \vee B), i\rangle\}$, $X_1 = X \cup \{\langle A, i\rangle\}$ and $X_2 = X \cup \{\langle B, i\rangle\}$. Assume that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X. Since model $\mathfrak{M}_{\mathbf{S5}} = \langle W, Q, V, w\rangle$ is appropriate for set X, there exists such function $f : \mathbb{N} \longrightarrow W$ that $\langle W, Q, V, f(i)\rangle \vDash (A \vee B)$. Therefore, by definition of truth in model 4.7, $\langle W, Q, V, f(i)\rangle \vDash A$ or $\langle W, Q, V, f(i)\rangle \vDash B$. Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1 or X_2, it is appropriate for some set X_i, where $1 \leq i \leq n$.
3. Assume that $R = R_{\rightarrow}$ and $X = Y \cup \{\langle (A \rightarrow B), i\rangle\}$, $X_1 = X \cup \{\langle \neg A, i\rangle\}$ and $X_2 = X \cup \{\langle B, i\rangle\}$. Assume that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X. Since model $\mathfrak{M}_{\mathbf{S5}} = \langle W, Q, V, w\rangle$ is appropriate for set X, there exists such function $f : \mathbb{N} \longrightarrow W$ that $\langle W, Q, V, f(i)\rangle \vDash (A \rightarrow B)$. Therefore, by definition of truth in model 4.7, $\langle W, Q, V, f(i)\rangle \vDash \neg A$ or $\langle W, Q, V, f(i)\rangle \vDash B$. Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1 or X_2, it is appropriate for some set X_i, where $1 \leq i \leq n$.
4. Assume that $R = R_{\leftrightarrow}$ and $X = Y \cup \{\langle (A \leftrightarrow B), i\rangle\}$, $X_1 = X \cup \{\langle A, i\rangle, \langle B, i\rangle\}$ and $X_2 = X \cup \{\langle \neg A, i\rangle, \langle \neg B, i\rangle\}$. Assume that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X. Since model $\mathfrak{M}_{\mathbf{S5}} = \langle W, Q, V, w\rangle$ is appropriate for set X, there exists such function $f : \mathbb{N} \longrightarrow W$ that $\langle W, Q, V, f(i)\rangle \vDash (A \leftrightarrow B)$. Therefore, by definition of truth in model 4.7, $\langle W, Q, V, f(i)\rangle \vDash A$ and $\langle W, Q, V, f(i)\rangle \vDash B$ or $\langle W, Q, V, f(i)\rangle \vDash \neg A$ and $\langle W, Q, V, f(i)\rangle \vDash \neg B$. Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1 or X_2, it is appropriate for some set X_i, where $1 \leq i \leq n$.
5. Assume that $R = R_{\neg\neg}$ and $X = Y \cup \{\langle \neg\neg A, i\rangle\}$, $X_1 = X \cup \{\langle A, i\rangle\}$. Assume that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X. Since model $\mathfrak{M}_{\mathbf{S5}} = \langle W, Q, V, w\rangle$ is appropriate for set X, there exists such function $f : \mathbb{N} \longrightarrow W$ that $\langle W, Q, V, f(i)\rangle \vDash \neg\neg A$. Therefore, by definition of truth in model 4.7, $\langle W, Q, V, f(i)\rangle \nvDash \neg A$, and consequently $\langle W, Q, V, f(i)\rangle \vDash A$. Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1, it is appropriate for some set X_i, where $1 \leq i \leq n$.
6. Assume that $R = R_{\neg\wedge}$ and $X = Y \cup \{\langle \neg(A \wedge B), i\rangle\}$, $X_1 = X \cup \{\langle \neg A, i\rangle\}$ and $X_2 = X \cup \{\langle \neg B, i\rangle\}$. Assume that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X. Since

model $\mathfrak{M}_{\mathbf{S5}} = \langle W, Q, V, w\rangle$ is appropriate for set X, there exists such function $f : \mathbb{N} \longrightarrow W$ that $\langle W, Q, V, f(i)\rangle \vDash \neg(A \wedge B)$. Therefore, by definition of truth in model 4.7, $\langle W, Q, V, f(i)\rangle \nvDash A$ or $\langle W, Q, V, f(i)\rangle \nvDash B$, so $\langle W, Q, V, f(i)\rangle \vDash \neg A$ or $\langle W, Q, V, f(i)\rangle \vDash \neg B$. Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1 or X_2, it is appropriate for some set X_i, where $1 \leq i \leq n$.

7. Assume that $R = R_{\neg\vee}$ and $X = Y \cup \{\langle\neg(A \vee B), i\rangle\}$, $X_1 = X \cup \{\langle\neg A, i\rangle, \langle\neg B, i\rangle\}$. Assume that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X. Since model $\mathfrak{M}_{\mathbf{S5}} = \langle W, Q, V, w\rangle$ is appropriate for set X, there exists such function $f : \mathbb{N} \longrightarrow W$ that $\langle W, Q, V, f(i)\rangle \vDash \neg(A \vee B)$. Therefore, by definition of truth in model 4.7, $\langle W, Q, V, f(i)\rangle \nvDash A$ and $\langle W, Q, V, f(i)\rangle \nvDash B$, and consequently $\langle W, Q, V, f(i)\rangle \vDash \neg A$ and $\langle W, Q, V, f(i)\rangle \vDash \neg B$. Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1, it is appropriate for some set X_i, where $1 \leq i \leq n$.
8. Assume that $R = R_{\neg\rightarrow}$ and $X = Y \cup \{\langle\neg(A \rightarrow B), i\rangle\}$, $X_1 = X \cup \{\langle A, i\rangle, \langle\neg B, i\rangle\}$. Assume that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X. Since model $\mathfrak{M}_{\mathbf{S5}} = \langle W, Q, V, w\rangle$ is appropriate for set X, there exists such function $f : \mathbb{N} \longrightarrow W$ that $\langle W, Q, V, f(i)\rangle \vDash \neg(A \rightarrow B)$. Therefore, by definition of truth in model 4.7, we get: $\langle W, Q, V, f(i)\rangle \nvDash (A \rightarrow B)$, and thus $\langle W, Q, V, f(i)\rangle \vDash A$ and $\langle W, Q, V, f(i)\rangle \nvDash B$, hence $\langle W, Q, V, f(i)\rangle \vDash \neg B$. Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1, it is appropriate for some set X_i, where $1 \leq i \leq n$.
9. Assume that $R = R_{\neg\leftrightarrow}$ and $X = Y \cup \{\langle\neg(A \leftrightarrow B), i\rangle\}$, $X_1 = X \cup \{\langle\neg A, i\rangle, \langle B, i\rangle\}$ and $X_2 = X \cup \{\langle A, i\rangle, \langle\neg B, i\rangle\}$. Assume that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X. Since model $\mathfrak{M}_{\mathbf{S5}} = \langle W, Q, V, w\rangle$ is appropriate for set X, there exists such function $f : \mathbb{N} \longrightarrow W$ that $\langle W, Q, V, f(i)\rangle \vDash \neg(A \leftrightarrow B)$. Therefore, by definition of truth in model 4.7, either $\langle W, Q, V, f(i)\rangle \nvDash A$ and $\langle W, Q, V, f(i)\rangle \vDash B$, or $\langle W, Q, V, f(i)\rangle \vDash A$ and $\langle W, Q, V, f(i)\rangle \nvDash B$, and thus $\langle W, Q, V, f(i)\rangle \vDash \neg A$ and $\langle W, Q, V, f(i)\rangle \vDash B$ or $\langle W, Q, V, f(i)\rangle \vDash A$ and $\langle W, Q, V, f(i)\rangle \vDash \neg B$. Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1 or X_2, it is appropriate for some set X_i, where $1 \leq i \leq n$.
10. Assume that $R = R_{\neg\Box}$ and $X = Y \cup \{\langle\neg\Box A, i\rangle\}$, $X_1 = X \cup \{\langle\Diamond\neg A, i\rangle\}$. Assume that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X. Since model $\mathfrak{M}_{\mathbf{S5}} = \langle W, Q, V, w\rangle$ is appropriate for set X, there exists such function $f : \mathbb{N} \longrightarrow W$ that $\langle W, Q, V, f(i)\rangle \vDash \neg\Box A$. Therefore, by definition of truth in model 4.7, $\langle W, Q, V, f(i)\rangle \nvDash \Box A$, and consequently there exists such $u \in W$ that $f(i)Qu$ & $\langle W, Q, V, u\rangle \nvDash A$, so $f(i)Qu \,\&\, \langle W, Q, V, u\rangle \vDash \neg A$, and thus $\langle W, Q, V, f(i)\rangle \vDash \Diamond\neg A$. Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1, it is appropriate for some set X_i, where $1 \leq i \leq n$.
11. Assume that $R = R_{\neg\Diamond}$ and $X = Y \cup \{\langle\neg\Diamond A, i\rangle\}$, $X_1 = X \cup \{\langle\Box\neg A, i\rangle\}$. Assume that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X. Since model $\mathfrak{M}_{\mathbf{S5}} = \langle W, Q, V, w\rangle$ is appropriate for set X, there exists such function $f : \mathbb{N} \longrightarrow W$

that $\langle W,Q,V,f(i)\rangle \vDash \neg \Diamond A$. Therefore, by definition of truth in model 4.7, $\langle W,Q,V,f(i)\rangle \nvDash \Diamond A$, and consequently there is no such $u \in W$ that $f(i)Qu \& \langle W,Q,V,u\rangle \vDash A$, so for any $u \in W$, if $f(i)Qu$, then $\langle W,Q,V, u\rangle \nvDash A$, that is for any $u \in W$, if $f(i)Qu$, then $\langle W,Q,V,u\rangle \vDash \neg A$, and thus $\langle W,Q,V,f(i)\rangle \vDash \Box\neg A$. Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1, it is also appropriate for some set X_i, where $1 \leq i \leq n$.

12. Assume that $R = R_{\Box}$ and $X = Y \cup \{\langle \Box A,i\rangle, irj\}$, $X_1 = X \cup \{\langle A,j\rangle\}$. Assume that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X. Since model $\mathfrak{M}_{\mathbf{S5}} = \langle W,Q,V,w\rangle$ is appropriate for set X, there exists such function $f : \mathbb{N} \longrightarrow W$ that $\langle W,Q,V,f(i)\rangle \vDash \Box A$ and $f(i)Qf(j)$. Therefore, by definition of truth in model 4.7, for any $u \in W$, if $f(i)Qu$, then $\langle W,Q,V,u\rangle \vDash A$, so $\langle W,Q,V,f(j)\rangle \vDash A$. Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1, thus it is appropriate for some set X_i, where $1 \leq i \leq n$.
13. Assume that $R = R_{\Diamond}$ and $X = Y \cup \{\langle \Diamond A,i\rangle\}$, $X_1 = X \cup \{irj, \langle A,j\rangle\}$. Assume that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X. Since model $\mathfrak{M}_{\mathbf{S5}} = \langle W,Q,V,w\rangle$ is appropriate for set X, there exists such function $f : \mathbb{N} \longrightarrow W$ that $\langle W,Q,V,f(i)\rangle \vDash \Diamond A$. Therefore, by definition of truth in model 4.7, there exists such $u \in W$ that $f(i)Qu \& \langle W,Q,V,u\rangle \vDash A$. By definition of rule $R_{\Diamond}$, index $j \notin *(X)$. So, we define new function $f' : \mathbb{N} \longrightarrow W$, so that for any $k \neq j$, $f'(k) = f(k)$, whereas $f'(j) = u$. Then, for any $k \neq j$ and $l \neq j$, if $\langle A,k\rangle \in X_1$, then $\langle W,Q,V,f'(k)\rangle \vDash A$, and if $krl \in X_1$, then $f'(k)Qf'(l)$. Moreover, $f'(i)Qf'(j) \& \langle W,Q,V,f'(j)\rangle \vDash A$, since $f'(j) = u$. Hence, from the above and from definition of model appropriate for the set of expressions 4.20, model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X_1. While since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1, it is also appropriate for some set X_i, where $1 \leq i \leq n$.
14. Assume that $R = R_r$ and $X_1 = X \cup \{irj\}$, where i, $j \in *(X)$. Assume that model $\mathfrak{M}_{\mathbf{S5}} = \langle W,Q,V,w\rangle$ is appropriate for set X, thus there exists function $f : \mathbb{N} \longrightarrow W$ that meets the conditions from definition of appropriate model 4.20 and $f(i), f(j) \in W$. Since relation R in model $\mathfrak{M}_{\mathbf{S5}} = \langle W,Q,V,w\rangle$ is universal, $f(i)Qf(j)$. Thus model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1, thus it is also appropriate for some set X_i, where $1 \leq i \leq n$. □

The above fact 4.77 will be used for the proof of another lemma. This lemma determines the relationship between the finite sets of formulas and the existence of maximal and open branches.

Lemma 4.78. *Let $X \subseteq \mathsf{For}_{\mathbf{S5}}$ be a finite set of formulas, $i \in \mathbb{N}$ and let $\mathfrak{M}_{\mathbf{S5}}$ be a model. If $\mathfrak{M}_{\mathbf{S5}} \vDash X$, then there exists a maximal and open branch beginning with set* $\{\langle A,i\rangle : A \in X\}$.

Proof. Take any finite set of formulas $X \subseteq \mathsf{For}_{\mathbf{S5}}$, any index $i \in \mathbb{N}$ and any model $\mathfrak{M}_{\mathbf{S5}} = \langle W, Q, V, w \rangle$, and then assume that $\mathfrak{M}_{\mathbf{S5}} \models X$. Let us define set $\{\langle A, i \rangle : A \in X\}$. Set $\{\langle A, i \rangle : A \in X\}$ will be denoted as X^i.

Now, we take any function $f : \mathbb{N} \longrightarrow W$ such that $f(i) = w$. By definition of model appropriate for the set of expressions 4.20, model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X^i, since for any formula $A \in \mathsf{For}_{\mathbf{S5}}$ and any index $j \in \mathbb{N}$, if $\langle A, j \rangle \in X^i$, then $j = i$. While due to the fact that $f(i) = w$ and the assumption that $\mathfrak{M}_{\mathbf{S5}} \models X$, we get a constitution that if $\langle A, i \rangle \in X^i$, then $\langle W, Q, V, f(i) \rangle \models A$. Moreover no expression krl, where $k, l \in \mathbb{N}$, belongs to set X^i, so there also holds the second condition of definition of model appropriate for the set of expressions 4.20.

Now, indirectly assume that each maximal branch beginning with set $X^i = \{\langle A, i \rangle : A \in X\}$ is closed.

As $\Phi(X^i)$ we will denote the set of all maximal branches beginning with set X^i. From fact 4.74, we know that for each finite set of tableau expressions Y there exists a maximal branch beginning with set Y. Thus, set $\Phi(X^i)$ is non-empty.

Since set $\Phi(X^i)$ is a set of all maximal branches beginning with set X^i, so it also has the following property. Assume that it contains branch χ. Let for certain $n \in \mathbb{N}$ exist such rule $R \in \mathbf{R}_{\mathbf{S5}}$ and such triple $\langle Z_1, Z_2, Z_3 \rangle \in R$ that $\chi(n) = Z_1$ and $\chi(n+1) = Z_2$ or $\chi(n+1) = Z_3$. Note that then both Z_2 and Z_3 are finite sets of expressions because each rule extends set to add at most two tableau expressions (by definition of tableau rules 4.23), branch χ begins with finite set X^i and we consider its n-th element. Thus, from fact 4.74 we know that:

- there exists maximal branch ϕ beginning with set X^i such that $\phi(n+1) = Z_2$
- there exists maximal branch ψ beginning with set X^i such that $\psi(n+1) = Z_3$.

$(*)$ Thus, for any $n \in \mathbb{N}$, if there exist: such rule $R \in \mathbf{R}_{\mathbf{S5}}$, and such l-tuple $\langle Z_1, \ldots, Z_l \rangle \in R$, where $1 < l \leq 3$, branch $\chi \in \Phi(X^i)$ such that $\chi(n) = Z_1$ and $\chi(n+1) = Z_2$ or $\chi(n+1) = Z_3$, then there exist branches $\psi \in \Phi(X^i)$ and $\phi \in \Phi(X^i)$ such that $\psi(n) = Z_1$, $\psi(n+1) = Z_2$ and $\phi(n) = Z_1$, $\phi(n+1) = Z_3$, if $l = 3$.

$(**)$ By assumption, each branch that belongs to set $\Phi(X^i)$ is closed, thus by fact 4.31, each branch that belongs to set $\Phi(X^i)$ has a finite length of m, for some $m \in \mathbb{N}$.

From the initial assumption, we know that each of branches in set $\Phi(X^i)$ begins with set X^i.

Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set of expressions X^i, by virtue of fact 4.21, which says that for a t-inconsistent set of expressions there is no appropriate model, set X^i is not t-inconsistent. Hence, we have a conclusion that in set $\Phi(X^i)$ there is no branch of length of one as a one long branch would be open.

Due to fact 4.77 which states that for any rule $R \in \mathbf{R_{S5}}$ any l-tuple — where $2 \le l \le 3$ — $\langle Z_1, \dots Z_l \rangle \in R$, if model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set Z_1, then it is appropriate for some set Z_i, where $2 \le i \le 3$, and $(*)$, there exists branch $\chi \in \Phi(X^i)$ such that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set $\chi(2)$ and $\chi(1) = X^i$. The set of those branches that belong to $\Phi(X^i)$, and at the same time model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for their k-th element, will be denoted as $\Phi(X^i)_k$. So, we have $\Phi(X^i) = \Phi(X^i)_1 \supseteq \Phi(X^i)_2 \ne \varnothing$.

Now, assume that for some $n \in \mathbb{N}$, where $n > 1$, set $\Phi(X^i)_{n-1} \supseteq \Phi(X^i)_n \ne \varnothing$. Since set $\Phi(X^i)_n$ is non-empty, so take some branch $\psi \in \Phi(X^i)_n$.

By assumption, model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set of expressions $\psi(n)$. Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set of expressions $\psi(n)$, so by virtue of fact 4.21, which says that for a t-inconsistent set of expressions there is no appropriate model, set of expressions $\psi(n)$ is not t-inconsistent. Thus, branch ψ is longer that n because otherwise it would be an open branch. Due to fact 4.77 which states that for any rule $R \in \mathbf{R_{S5}}$ any l-tuple — where $2 \le l \le 3$ — $\langle Z_1, \dots Z_l \rangle \in R$, if model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set Z_1, then it is appropriate for some set Z_i, where $2 \le i \le 3$, and $(*)$, there exists branch $\phi \in \Phi(X^i)_1$ such that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set $\phi(n+1)$ and $\phi \in \Phi(X^i)_n$. Thus, $\Phi(X^i)_n \supseteq \Phi(X^i)_{n+1}$ and $\Phi(X^i)_{n+1} \ne \varnothing$.

Therefore, for each $k \in \mathbb{N}$, $\Phi(X^i)_k \ne \varnothing$ and

$$\Phi(X^i) = \Phi(X^i)_1 \supseteq \Phi(X^i)_2 \supseteq \dots \supseteq \Phi(X^i)_k \supseteq \dots$$

Next, we take the intersection of all those sets $\Phi(X^i)_k$, where $k \in \mathbb{N}$. Intersection $\bigcap\{\Phi(X^i)_k : k \in \mathbb{N}\} = \Phi$ is non-empty as for each k, subset $\Phi(X^i)_k$ is also non-empty. So, set Φ includes at least one branch χ. That branch is maximal and begins with set X^i since $\Phi \subseteq \Phi(X^i)$.

But, branch χ is infinite which contradicts conclusion $(**)$. □

We can now move on to the last lemma of this chapter.

Lemma 4.79. *For any $X \subseteq \mathsf{For}_{\mathbf{S5}}$, $A \in \mathsf{For}_{\mathbf{S5}}$, if there exists finite set $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi \rangle$, then $X \vDash A$.*

Proof. Take any $X \subseteq \mathsf{For}_{\mathbf{S5}}$ and $A \in \mathsf{For}_{\mathbf{S5}}$. Assume that $X \nvDash A$. So, by definition of relation of semantic consequence 4.9, there exists such model $\mathfrak{M}_{\mathbf{S5}}$ that $\mathfrak{M}_{\mathbf{S5}} \vDash X$ and $\mathfrak{M}_{\mathbf{S5}} \nvDash A$. Therefore, by definition of truth in model 4.7, we have $\mathfrak{M}_{\mathbf{S5}} \vDash \neg A$, and consequently $\mathfrak{M}_{\mathbf{S5}} \vDash X \cup \{\neg A\}$. Thus for any finite set $Y \subseteq X$, also $\mathfrak{M}_{\mathbf{S5}} \vDash Y \cup \{\neg A\}$.

Take any finite subset $Y' \subseteq X$. From lemma 4.78 we get a conclusion that for any $i \in \mathbb{N}$ there exists maximal and open branch beginning with set $\{\langle B, i \rangle : B \in Y' \cup \{\neg A\}\}$. And from the above, and from lemma 4.76 we know that each complete tableau $\langle Y', A, \Phi\rangle$ is open. Since Y' was an arbitrary finite subset of set of formulas X, so there is no finite set $Y \subseteq X$ and closed tableau $\langle Y', A, \Phi\rangle$. □

To sum up the lemmas we have presented so far, we move on to the theorem on completeness for the tableau system we have discussed.

Theorem 4.80 (Theorem on the completeness of tableau system of **S5**). *For any $X \subseteq \mathit{For}_{S5}$, $A \in \mathit{For}_{S5}$, the below statements are equivalent.*

- $X \vDash A$
- $X \triangleright A$
- *there exists finite set $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi\rangle$*

Proof. Take any $X \subseteq \mathsf{For}_{\mathbf{S5}}$ and $A \in \mathsf{For}_{\mathbf{S5}}$. For the theorem proof, it is sufficient to show the occurrence of three implications:

(a) $X \vDash A \Rightarrow X \triangleright A$
(b) $X \triangleright A \Rightarrow$ there exists finite $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi\rangle$
(c) there exists finite set $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi\rangle \Rightarrow X \vDash A$.

Implication (a) results from lemma 4.59. Implication (b) results from lemma 4.75. Implication (c) results from lemma 4.79. □