5 Metatheory of tableau systems for propositional logic and term logic

5.1 Introductory remarks

In this chapter, we establish general tableau concepts for systems constructed using the method described in the book. These concepts make it possible to utter and justify a number of basic facts, the specific cases of which we have been proving in the previous chapters.

Using the further defined general concepts of the tableau systems, we will be able to utter and prove a general theorem on the relationship between the tableau systems and the semantics adopted for them. The construction of the tableau system, which is adequate in terms of the adopted semantics, will boil down to defining the basic concepts of this system in such a way that they are special cases of the general concepts and meet certain general conditions which we will define further. In this way, we will shorten the definition of the tableau systems to a minimal — in comparison to the previous chapters — number of procedures that describe the features of the considered system that distinguish it from other tableau systems.

5.2 Language and semantics

By set For we mean any set of formulas of some language. We call its elements *formulas*.

Remark 5.1. For our considerations, we adopt any but fixed such set of formulas For that $|\mathsf{For}|$ is an even number or For is an infinite set. Set For will remain unchanged until the end of this chapter.

We will now look at the issue of interpretation of formulas. In the cases presented in the previous chapters, the interpretations were the valuations of formulas or models. However, we will deliberately use the concept of *interpretation* in order to cover all those cases. We intend to describe in general the concept of interpretation which will be applicable in the presented metatheory of the construction of tableau systems that will allow us to draw conclusions on the general relationships between the semantic form of a given logic and its tableau approach.

In our considerations, we will make use of the fact that we only look into such logics whose interpretations assign exactly one of the two logical values to each formula. So, a given interpretation divides a set of formulas in an unambiguous, disjoint and exhaustive way into a subset of true formulas and a subset of false formulas under this interpretation. This division coincides with some division of the set of formulas into mutually contradictory formulas, because for any formula, a formula is true in a given interpretation if and only if the formula contradicting it is false in that interpretation.

The interpretation of a set of formulas can therefore be identified with the segment of the logical division of a set of formulas in terms of the contradiction of formulas which contains exactly all the formulas that are true in this interpretation. The starting point, however, will be a function that assigns a contradictory formula to each formula, thus always dividing the set of formulas into two segments of the logical division. At least one of those segments may correspond to some interpretation of a set of formulas in which all and only those formulas that belong to this segment of the division are true. We will illustrate this with an example.

Example 5.2. Let us consider the case of **TL** discussed in Chapter Three. Function $\circ$ (definition 3.12) assigns a contradictory formula to each formula from set $\mathsf{For}_{\mathbf{TL}}$ (fact 3.13). Let us now divide set of formulas $\mathsf{For}_{\mathbf{TL}}$ into the following pairs of sets: X' and complement of X' to set of formulas $\mathsf{For}_{\mathbf{TL}}$, i.e. $X'' = \mathsf{For}_{\mathbf{TL}} \setminus X'$, as per the principle of division, for each formula $A \in \mathsf{For}_{\mathbf{TL}}$:

$$(\dagger)\ A \in X' \text{ iff } \circ(A) \notin X'.$$

The division of set of formulas $\mathsf{For}_{\mathbf{TL}}$ into sets X' and X'' is a logical division as $X' \cap X'' = \varnothing$ and $X' \cup X'' = \mathsf{For}_{\mathbf{TL}}$, and what is more, sets X' and X'' are non-empty.

Let us denote the set of all models $\mathfrak{M}_{\mathbf{TL}}$ for **TL** as $\mathbf{M}_{\mathbf{TL}}$, while the set of all such segments of division by function $\circ$ that meet equivalence $(\dagger)$ as $\mathbf{X}_{\circ}$.

Let $\mathfrak{M}_{\mathbf{TL}}$ be any model. We define subset of formulas $\mathfrak{M}'_{\mathbf{TL}} = \{A \in \mathsf{For}_{\mathbf{TL}} : \mathfrak{M}_{\mathbf{TL}} \vDash A\}$. A complement of set $\mathfrak{M}'_{\mathbf{TL}}$ to set $\mathsf{For}_{\mathbf{TL}}$ is set $\mathsf{For}_{\mathbf{TL}} \setminus \mathfrak{M}'_{\mathbf{TL}}$, i.e. set $\mathfrak{M}''_{\mathbf{TL}} = \{A \in \mathsf{For}_{\mathbf{TL}} : \mathfrak{M}_{\mathbf{TL}} \nvDash A\}$.

Function $\circ$ assigns a contradictory formula to each formula from set $\mathsf{For}_{\mathbf{TL}}$ (fact 3.13), since for any model $\mathfrak{M}_{\mathbf{TL}}$ for **TL** and for any formula $A \in \mathsf{For}_{\mathbf{TL}}$ it is the case that:

$$(\dagger\dagger)\ \mathfrak{M}_{\mathbf{TL}} \vDash A \text{ iff } \mathfrak{M}_{\mathbf{TL}} \nvDash \circ(A).$$

Since sets $\mathfrak{M}'_{\mathbf{TL}}$ and $\mathfrak{M}''_{\mathbf{TL}}$ are disjoint, exhaustive and non-empty, so they make up one of many logical divisions of set $\mathsf{For}_{\mathbf{TL}}$ by function $\circ$ according to principle $(\dagger)$, for any formula $A \in \mathsf{For}_{\mathbf{TL}}$:

$$(\dagger\dagger\dagger)\ A \in \mathfrak{M}'_{\mathbf{TL}} \text{ iff } \circ(A) \notin \mathfrak{M}'_{\mathbf{TL}}.$$

and belong to set $\mathbf{X}_{\circ}$.

So with function $\circ$ we can unambiguously identify each model $\mathfrak{M}_{\mathbf{TL}}$ (and more precisely the set of all formulas that are true in this model) with some segment of the logical division of the set of formulas by the contradiction of formulas, determined by function $\circ$ with equivalence $(\dagger\dagger)$.

For each $\mathfrak{M}_{\mathbf{TL}} \in \mathbf{M}_{\mathbf{TL}}$, there exists precisely one set $X \in \mathbf{X}_{\circ}$ such that $\mathfrak{M}'_{\mathbf{TL}} = X$. Therefore, we can identify set of all models $\mathbf{M}_{\mathbf{TL}}$ with some subset of set $\mathbf{X}_{\circ}$, that is with some subset of set of all subsets of set of formulas $P(\mathsf{For}_{\mathbf{TL}})$.

However, the opposite dependence does not occur. The segments of logical division that belong to $\mathbf{X}_{\circ}$ may correspond to many models that vary in terms of the domain cardinality, but not formulas which are true in them.

In some cases, a segment of logical division that belongs to $\mathbf{X}_{\circ}$ may not be determined by any models. If, for instance, we take such segments of division Y' and Y'' that belong to $\mathbf{X}_{\circ}$ that for certain name letters P, $Q \in \mathsf{Ln}$, $P\mathbf{e}P \in Y'$ and $P\mathbf{i}Q \in Y'$, then there is no such model $\mathfrak{M}_{\mathbf{TL}} \in \mathbf{M}_{\mathbf{TL}}$ that $\mathfrak{M}_{\mathbf{TL}} \models Y'$, since the truth of set of formulas Y' would require the denotation of letter P to be an empty set and non-empty one at the same time. Further, we will also provide an example for **CPL** (example 5.6).

Therefore, models/valuations, that is the set of interpretations should be identified with a certain subset of the set of logical divisions of the set of formulas determined by a certain function that assigns a contradictory formula to each formula. Whether or not it is a proper subset, may vary from one case to another (see example 5.7).

In the definition of interpretation of formulas, we will employ a reference to the sets of formulas that are true in a given model or valuation, because in the general approach to the tableau metatheory we cannot penetrate the structure of particular types of interpretations for different logics. At the same time, we must retain the general aspects, in particular those that correspond to the tableau proof, i.e. adopting in the proof a formula that is contradictory to the formula we are proving.

We will now propose the general definition of set of divisions of set of formulas For. There is a function in this definition that implicitly assigns a contradictory formula to each formula. In many cases, however, even if this function is well established, we will identify a set of initial valuations/models with a proper subset of some sort of a set of all divisions.

Definition 5.3. Let $f : \mathsf{For} \longrightarrow \mathsf{For}$ be an injective function and $X \subseteq \mathsf{For}$. We shall state that X is a *division* of For iff for any formula $A \in \mathsf{For}$ the following condition is met: $A \in X$ iff $f(A) \notin X$.

By set $\mathbf{X}_f$ we will mean a set of all divisions of set of formulas For determined by the established function f.

Remark 5.4. In some cases, function f may result in the non-existence of any division $X \subseteq \mathsf{For}$, and then, consequently, set of all divisions $\mathbf{X}_f$ is empty. It is the case for instance when for some formula $A \in \mathsf{For}$ $f(A) = A$. Thus, not each function f is suitable for the definition of set $\mathbf{X}_f$.

In some cases, such function may fail to determine divisions corresponding to all models/valuations (example 5.7). For these cases, the specified method cannot be used to define a tableau system.

Function f is determined by default by the way of adopting an equivalent of formula contradictory to the formula being proved. So, function f can correspond to negation — it does in the case of **CPL** and **S5**[1]. Moreover, it may not correspond to any functor from the language — as it does not in the event of the tableau system for **TL**, where it could be identified with function $\circ$ 3.13 (example 5.2) that assigns a contradictory formula to each formula. Generally speaking, function f, to each formula, assigns such formula that for any division X it is the case that exactly one of those formulas belongs to X.

Remark 5.5. For our considerations, we adopt any but fixed and non-empty set of all divisions $\mathbf{X}_f$, for some function $f : \mathsf{For} \longrightarrow \mathsf{For}$ that meets the conditions of the last definition 5.3. The set will remain unchanged until the end of this chapter. We will, on the other hand, refer to function f.

As stated in example 5.2, for a given set of all valuations/models and given function f, it does not have to be the case that each element of set $\mathbf{X}_f$ corresponds to some valuation/model. Below, we provide an example for **CPL** (example 5.6).

Example 5.6. Take set of formulas of **CPL** $\mathsf{For}_{\mathbf{CPL}}$. We will define function f as follows $f(A) = \neg A$, for any formula $A \in \mathsf{For}_{\mathbf{CPL}}$. Function f meets the condition from definition of division of set of formulas 5.8, i.e. it is an injective function. Now, let $\mathbf{X}_f$ be a set of all divisions of set of formulas determined by function f.

The set contains such division X that $(p \wedge q) \in X$, $\neg p \in X$ and $\neg q \in X$. Obviously, no valuation of formulas V corresponds to set X, since for each V, if $V((p \wedge q)) = 1$, then $V(\neg p) = 0$. Therefore, there does not exist such valuation V that $V' = \{A \in \mathsf{For} : V(A) = 1\}$ and $V' = X$.

Moreover, no valuation corresponds to set $\mathsf{For}_{\mathbf{CPL}} \setminus X$, since for each V, if $V(\neg(p \wedge q)) = 1$, then $V(\neg\neg p) = 0$ or $V(\neg\neg q) = 0$, and $V(p) = 0$ or $V(q) = 0$. Therefore, there does not exist such valuation V that $V' = \{A \in \mathsf{For} : V(A) = 1\}$ and $V' = \mathsf{For}_{\mathbf{CPL}} \setminus X$.

1 However, function f does not have to imply the addition of negation to the formula. For we could, for instance, assign a contradictory formula to each formula of **CPL** which would not be a negation of the initial formula. In the same way, we could also define a complete tableau system.

On the other hand, however, for each valuation V there exists such set $V' = \{A \in \mathsf{For} : V(A) = 1\}$ that for some division $Y \in \mathbf{X}_f$, $V' = Y$, because for each formula $A \in \mathsf{For}_{\mathbf{CPL}}$ it is the case that $A \in V'$ iff $\neg A \notin V'$.

By virtue of the above, we can identify the set of all valuations of formulas with a proper subset of set of all divisions $\mathbf{X}_f$ by function f.

In some cases and for some functions f it may be the case that the set of divisions corresponding to all models/valuations is identical to given set $\mathbf{X}_f$ (example 5.7).

Example 5.7. Consider two similar cases. In the first one, function f cannot be established in such a way that each model has a corresponding division that determines f. In the second case, for each division by function f there exists at least one corresponding model.

Take such subset X of set of all formulas $\mathsf{For}_{\mathbf{TL}}$ that $X = \{P^1\mathbf{a}Q^1, P^1\mathbf{e}Q^1\}$. Consider set of all models $\mathbf{M}_X$ for a new, more sparing language. They will be analogous to the models for **TL**, but their denotation sets will only be assigned to two name letters that occur in a new, more sparing language — letters P^1 and Q^1.

Next, in a usual manner we will define the relation of semantic consequence determined on set $P(X) \times X$ (in a sense truncating the relation of semantic consequence of **TL** to set X) and denote it as $\models'$. Only four arguments are correct in such logic:

$$\{P^1\mathbf{a}Q^1, P^1\mathbf{e}Q^1\} \models' P^1\mathbf{a}Q^1$$

$$\{P^1\mathbf{a}Q^1, P^1\mathbf{e}Q^1\} \models' P^1\mathbf{e}Q^1$$

$$\{P^1\mathbf{a}Q^1\} \models' P^1\mathbf{a}Q^1$$

$$\{P^1\mathbf{e}Q^1\} \models' P^1\mathbf{e}Q^1$$

since:

$$\{P^1\mathbf{a}Q^1\} \not\models' P^1\mathbf{e}Q^1$$

$$\{P^1\mathbf{e}Q^1\} \not\models' P^1\mathbf{a}Q^1$$

due to the fact that in $\mathbf{M}_X$ there exist models in which the denotation of letters P^1 is a non-empty set. In the event when that set is contained in the set of denotations Q^1 proposition $P^1\mathbf{a}Q^1$ is true, while proposition $P^1\mathbf{e}Q^1$ is not true. Instead, in the event when the set of denotations of name letter P^1 is disjoint with the set of denotations of letter Q^1, proposition $P^1\mathbf{e}Q^1$ is true, while proposition $P^1\mathbf{a}Q^1$ is not true.

In turn, due to the fact that in $\mathbf{M}_X$ there exist models in which the denotation of letter P^1 is an empty set, set $X = \{P^1\mathbf{a}Q^1, P^1\mathbf{e}Q^1\}$ is not a contradictory set. Therefore, we are unable to establish function f in such a way that each model has a division of set X determined by function f. For there exists only one injective function $f : X \longrightarrow X$ such that for any $y \in X$ $f(y) \neq y$. It is defined as follows:

$$f(P^1\mathbf{a}Q^1) = P^1\mathbf{e}Q^1$$

$$f(P^1\mathbf{e}Q^1) = P^1\mathbf{a}Q^1.$$

Now, take any division Y of set X by function f. It meets condition: $A \in Y$ iff $f(A) \notin Y$, for any $A \in X$. If $P^1\mathbf{a}Q^1 \in Y$, then $f(P^1\mathbf{a}Q^1) \notin Y$, i.e. $P^1\mathbf{e}Q^1 \notin Y$, thus $P^1\mathbf{e}Q^1 \in X \setminus Y$. If, in turn, $P^1\mathbf{a}Q^1 \notin Y$, then $f(P^1\mathbf{a}Q^1) \in Y$, i.e. $P^1\mathbf{e}Q^1 \in Y$, thus $P^1\mathbf{e}Q^1 \notin X \setminus Y$. An analogous sequence of implications occurs for the other formula. Thus, for model in which set of formulas $X = \{P^1\mathbf{a}Q^1, P^1\mathbf{e}Q^1\}$ is true, there does not exist any corresponding division of set X by function f. So, for such relation of consequence like $\models'$, we will not construct a tableau system using the provided method[2].

On the other hand, take such subset Y of set of all formulas $\mathsf{For}_{\mathbf{TL}}$, that $Y = \{P^1\mathbf{a}Q^1, P^1\mathbf{o}Q^1\}$, and set of all models $\mathbf{M}_Y$ (which is identical to set $\mathbf{M}_X$, as we still have the same letters in the alphabet), and then define in a usual manner the relation of semantic consequence on set $P(Y) \times Y$ (in a sense truncating the relation of semantic consequence of **TL** to set Y) and denote it as $\models''$. In such logic, there occur analogous implications like for relation $\models'$. However, in $\mathbf{M}_Y$ there does not exist model such that set $Y = \{P^1\mathbf{a}Q^1, P^1\mathbf{o}Q^1\}$ is true in it, because formulas $P^1\mathbf{a}Q^1$, $P^1\mathbf{o}Q^1$ are contradictory. We establish function $f : Y \longrightarrow Y$ as follows: $f(P^1\mathbf{a}Q^1) = P^1\mathbf{o}Q^1$ and $f(P^1\mathbf{o}Q^1) = P^1\mathbf{a}Q^1$. We only get two divisions of set Y by function f, $Y' = \{P^1\mathbf{a}Q^1\}$ and $Y'' = \{P^1\mathbf{o}Q^1\}$ [3].

Note that for each model $\mathfrak{M}_Y$ there exists division Z of set Y by function f such that $\mathfrak{M}'_Y = \{A \in Y : \mathfrak{M}_Y \models'' A\} = Z$. On the other hand, for each division Z of set Y by function f there exists model $\mathfrak{M}_Y$ such that $\mathfrak{M}'_Y = \{A \in Y : \mathfrak{M}_Y \models'' A\} = Z$.

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2 Perhaps for relation $\models'$ it is impossible at all to define a tableau system in a standard way, as it is not the case that for each formula there exists a contradictory formula. We can, however, expand the provided method, requiring the adoption at the beginning of proof of suitable auxiliary expressions instead of the contradictory formula, when it does not exist. In the discussed case, for instance for formula $P^1\mathbf{a}Q^1$ it would be set $\{P^1_{+i}, Q^1_{-i}\}$, for some $i \in \mathbb{N}$.

3 The example shown is interesting insomuch that in the tableau system for it, the set of tableau rules could be empty. For each branch and tableau would be one-step, because having adopted a set of premises in the very first step, we would receive a t-inconsistent set, with the natural assumption that $\{A, f(A)\}$ is a t-inconsistent set.

For the reasons we mentioned, in further semantic considerations we will refer to some established set $\mathbf{I} \subseteq \mathbf{X}_f$. So, let us proceed to the definition.

Definition 5.8 (Interpretation of formulas). *A set of interpretations determined by function f* (for short: *interpretations*) is each subset $\mathbf{I} \subseteq \mathbf{X}_f$. The elements of set **I** will be called *interpretations of formulas* (or for short: *interpretations*) and denoted by letter $\mathfrak{I}$, possibly with indices.

Denotation 5.9. Let $\mathbf{I} \subseteq \mathbf{X}_f$. Let $\mathfrak{I} \in \mathbf{I}$ be any interpretation of formulas. Let us adopt the following denotations:

- for any formula $A \in \mathsf{For}$, the fact that $A \in \mathfrak{I}$ will be put as $\mathfrak{I} \vDash A$, whereas the fact that $A \notin \mathfrak{I}$ will be put as $\mathfrak{I} \nvDash A$
- for any subset of formulas $X \subseteq \mathsf{For}$, $\mathfrak{I} \vDash X$ iff for any formula $A \in X$, $\mathfrak{I} \vDash A$; while $\mathfrak{I} \nvDash X$ iff it is not the case that $\mathfrak{I} \vDash X$.

We will now proceed to the concept of relation of semantic consequence.

Definition 5.10 (Semantic consequence). Let $\mathbf{I} \subseteq \mathbf{X}_f$ be a set of interpretations. Let $X \subseteq \mathsf{For}$ and $A \in \mathsf{For}$.

- Formula *A follows from X under* **I** (for short: $X \vDash_{\mathbf{I}} A$) iff for any interpretation $\mathfrak{I} \in \mathbf{I}$, if $\mathfrak{I} \vDash X$, then $\mathfrak{I} \vDash A$. Whereas formula *A does not follow from X under* **I** (for short: $X \nvDash_{\mathbf{I}} A$) iff it is not the case that $X \vDash_{\mathbf{I}} A$.
- When set **I** is fixed, we apply notation $X \vDash A$ and respectively $X \nvDash A$.
- Relation $\vDash$ will be called a *relation of semantic consequence* (*defined by set of interpretations* **I**).

With an established relation $\vDash$ we can proceed to the concept of semantically defined logic.

Definition 5.11. Let set $\mathbf{I} \subseteq \mathbf{X}_f$ be a set of interpretations of formulas For. Pair $\langle \mathsf{For}, \vDash_{\mathbf{I}} \rangle$ will be called a *semantically defined logic*.

Another fact says that two relations of consequence defined on set of formulas For are identical iff they are defined with the same set of interpretations $\mathbf{I} \subseteq \mathbf{X}_f$, so when speaking of relation $\vDash$, we do not have to refer to set **I**, as it is unambiguously determined by relation $\vDash$.

Proposition 5.12. *Let $\mathbf{I}_1 \subseteq \mathbf{X}_f$ and $\mathbf{I}_2 \subseteq \mathbf{X}_f$ be sets of interpretations of formulas For. Let pairs $\langle \mathsf{For}, \vDash_{\mathbf{I}_1} \rangle$ and $\langle \mathsf{For}, \vDash_{\mathbf{I}_2} \rangle$ be semantically defined logics. In such case $\vDash_{\mathbf{I}_1} = \vDash_{\mathbf{I}_2}$ iff $\mathbf{I}_1 = \mathbf{I}_2$.*

Proof. Take any sets of interpretations of formulas $\mathbf{I}_1 \subseteq \mathbf{X}_f$, $\mathbf{I}_2 \subseteq \mathbf{X}_f$, and logics $\langle \mathsf{For}, \vDash_{\mathbf{I}_1} \rangle$ and $\langle \mathsf{For}, \vDash_{\mathbf{I}_2} \rangle$ they semantically defined. Assume that $\mathbf{I}_1 \neq \mathbf{I}_2$. We have two cases:

(1) there exists such interpretation $\mathfrak{I} \in \mathbf{I}_1$ that $\mathfrak{I} \notin \mathbf{I}_2$

or

(2) there exists such interpretation $\mathfrak{I} \in \mathbf{I}_2$ that $\mathfrak{I} \notin \mathbf{I}_1$.

Let us only consider case (1) as case (2) is analogous.

Now, take any such interpretation $\mathfrak{I} \in \mathbf{I}_1$ that $\mathfrak{I} \notin \mathbf{I}_2$.

Assume that $\mathbf{I}_2$ is an empty set. By definition of relation of semantic consequence 5.10, $\vDash_{\mathbf{I}_2} = P(\mathsf{For}) \times \mathsf{For}$. On the other hand, since $\mathfrak{I} \in \mathbf{I}_1$, so by definition 5.3, for certain formula A, $\mathfrak{I} \vDash A$ and $\mathfrak{I} \nvDash f(A)$, so by definition of relation of semantic consequence 5.10, $\{A\} \nvDash_{\mathbf{I}_1} f(A)$, thus $\vDash_{\mathbf{I}_1} \neq \vDash_{\mathbf{I}_2}$.

$(*)$ Now, assume that $\mathbf{I}_2$ is not an empty set.

Since $\mathfrak{I} \notin \mathbf{I}_2$, then by definition 5.3 for each interpretation $\mathfrak{I}' \in \mathbf{I}_2$ there exists such formula $B \in \mathsf{For}$ that:

(a) $\mathfrak{I}' \vDash B$ and $\mathfrak{I} \nvDash B$

or

(b) $\mathfrak{I}' \nvDash B$ and $\mathfrak{I} \vDash B$.

Take any interpretation $\mathfrak{I}' \in \mathbf{I}_2$ and such formula $B \in \mathsf{For}$ that there occurs case (a) $\mathfrak{I}' \vDash B$ and $\mathfrak{I} \nvDash B$. It is the case when and only when — by definition 5.3 — (a)' $\mathfrak{I}' \nvDash f(B)$ and $\mathfrak{I} \vDash f(B)$, thus there exists such formula $C \in \mathsf{For}$ that $f(B) = C$ and (a)" $\mathfrak{I}' \nvDash C$ and $\mathfrak{I} \vDash C$.

Now, we define set of formulas $X \subseteq \mathsf{For}$ as follows: for any formula A, $A \in X$ iff there exists such interpretation $\mathfrak{I}' \in \mathbf{I}_2$ that $\mathfrak{I}' \nvDash A$ and $\mathfrak{I} \vDash A$.

From $(*)$, (a)", (b) it follows that set X is non-empty and for each interpretation $\mathfrak{I}' \in \mathbf{I}_2$, there exists such formula $D \in X$ that $\mathfrak{I}' \nvDash D$. What is more, for each formula $D \in X$, $\mathfrak{I} \vDash D$, so $\mathfrak{I} \vDash X$.

Therefore, there does not exist such interpretation $\mathfrak{I}' \in \mathbf{I}_2$ that $\mathfrak{I}' \vDash X$. From the above and from definition of relation of semantic consequence 5.10, we get that $X \vDash_{\mathbf{I}_2} D$, for any formula $D \in \mathsf{For}$, since $\mathfrak{I}' \nvDash X$, for any interpretation of formulas $\mathfrak{I}' \in \mathbf{I}_2$.

While since $\mathfrak{I} \vDash X$, so $\mathfrak{I} \nvDash f(E)$, for each $E \in X$, by definition 5.3. From the above and from definition of semantic consequence 5.10, $X \nvDash_{\mathbf{I}_1} f(E)$, for certain $E \in X$. And at the same time $X \vDash_{\mathbf{I}_2} f(E)$, since $X \vDash_{\mathbf{I}_2} D$, for any formula $D \in \mathsf{For}$, thus $\vDash_{\mathbf{I}_1} \neq \vDash_{\mathbf{I}_2}$

Now, assume that $\mathbf{I}_1 = \mathbf{I}_2$. Then obviously $\vDash_{\mathbf{I}_1} = \vDash_{\mathbf{I}_2}$, since $\vDash_{\mathbf{I}_1} = \vDash_{\mathbf{I}_1}$. □

Remark 5.13. For our considerations, we adopt any, but fixed set of interpretations $\mathbf{I} \subseteq \mathbf{X}_f$. The set will remain unchanged until the end of this chapter. We will, on the other hand, refer to function f.

Remark 5.14. For our considerations, we adopt any, but fixed, semantically defined logic $\langle \mathsf{For}, \vDash \rangle$. Due to fact 5.12, since relation of consequence $\vDash$ determines exactly one set of interpretations of formulas $\mathbf{I} \subseteq \mathbf{X}_f$, we do not have to include a reference to set $\mathbf{I}$ in the notation. Logic $\langle \mathsf{For}, \vDash \rangle$ will remain unchanged until the end of this chapter. On the other hand, we will from time to time refer to the set of interpretations of formulas $\mathbf{I}$ determined by $\vDash$.

5.3 Basic concepts of the tableau system

Now, we will define the set of conditions that should be satisfied by the tableau expressions.

Definition 5.15 (Tableau expressions)**.** *The set of tableau expressions* will be called any set Te that meets the following conditions:

1. there exists such injective function $g : \mathsf{For} \longrightarrow P(\mathsf{Te})$ that for any formula $A \in \mathsf{For}$, $g(A)$ is a countable subset of set Te and for any formula $B \in \mathsf{For}$, if $A \neq B$, then $g(A) \cap g(B) = \varnothing$
2. there exists at least one distinguished and finite set $\mathsf{Te}' \subseteq \mathsf{Te}$ such that for any subset $X \subseteq \mathsf{For}$, if there exists interpretation of formulas $\mathfrak{I} \in \mathbf{I}$ such that $\mathfrak{I} \vDash X$, then $\mathsf{Te}' \nsubseteq \bigcup \{g(A) : A \in X\}$ (the set of all such distinguished sets will be specified as Te^{in}).

The elements of set Te will be called *tableau expressions* or simply *expressions*.

Making use of condition 2 of definition of set of tableau expressions 5.15, we will now introduce the general concept of t-inconsistent set (and concept of t-consistent set).

Definition 5.16. Let Te be a set of tableau expressions and let $\mathsf{Te}'' \subseteq \mathsf{Te}$. Set Te'' will be called *tableau inconsistent* (for short: *t-inconsistent*) iff there exists such $\mathsf{Te}' \in \mathsf{Te}^{in}$ that $\mathsf{Te}' \subseteq \mathsf{Te}''$. Set Te'' will be called *tableau consistent* (for short: *t-consistent*) iff Te'' is not t-inconsistent.

Remark 5.17. For any set of expressions Te we assume that the values of function $g : \mathsf{For} \longrightarrow P(\mathsf{Te})$ are sets of indexed elements of set Te, i.e. for any formula $A \in \mathsf{For}$, there exists such countable set $\{x^1, x^2, x^3, \dots\} \subseteq \mathsf{Te}$ that $g(A) = \{x^1, x^2, x^3, \dots\}$. At the same time, we do not assume that for any x^i, $x^j \in g(A)$, if $i \neq j$, then $x^i \neq x^j$. The numbers visible in the indices of tableau expressions $x^1, x^2, x^3, \dots$ will be called *indices*.

Remark 5.18. As we remember, in the case for **CPL** considered in the book, the set of expressions was identical to the set of formulas. In this case, we would adopt

$\mathsf{Te} = \{A^i : A \in \mathsf{For}_{\mathbf{CPL}}, i \in \mathbb{N}\}$[4]. In a more complex case, i.e. **TL**, set $\{A^i : A \in \mathsf{For}_{\mathbf{TL}}, i \in \mathbb{N}\} \subseteq \mathsf{Te}$. In both those cases, we associate a little artificially with each formula an infinite set of expressions that correspond to it, whereas for each formula A, if $g(A) = \{x^1, x^2, x^3, \dots\}$, then for any indices i, j we would have: $x^i = x^j$.

In the case of **S5** such association is more natural, as the elements of set along with indices $\{A^i : A \in \mathsf{For}_{\mathbf{S5}}, i \in \mathbb{N}\} \subseteq \mathsf{Te}$ would correspond to the ordered couples $\langle A, i\rangle$ we use in the proofs.

In both latter cases (**TL**, **S5**), in order to obtain entire set Te we would have to still define additional auxiliary expressions. Our definition does not specify what these expressions would be, but in order to get a complete tableau system, they would have to meet further conditions that we will provide. This reservation also applies to the concept of an inconsistent set of expressions, which in some cases could even contain one element. This is the case in those systems where the branch containing certain expressions (considered to be inconsistent) is closed with a special sign by means of an additional rule or rules, e.g. × (refer for example [23]) and only then the branch is considered closed.

Ultimately, however, the role of these expressions which, through function g represent formulas in a tableau proof, could be fulfilled by any other symbols, not structurally (graphically) related to formulas, yet meeting the conditions from definition of set of tableau expressions 5.15.

New denotations will be useful for further work.

Denotation 5.19. Let us adopt the following denotations:

- for any formula $A \in \mathsf{For}$ and any $i \in \mathbb{N}$, $A^i = x^i$ iff $x^i \in g(A)$
- for any subset $X \subseteq \mathsf{For}$, $X^i = \{A^i : A \in X\}$.

Remark 5.20. For our considerations, we adopt any, but fixed set of tableau expressions Te and included in Te at least one tableau contradictory set Te'. These arrangements will remain unchanged until the end of this chapter.

Remark 5.21. We also assume the option of inclusion in set Te of auxiliary expressions which correspond to expressions such as *irj* in case of the described tableau system for **S5**, or such as expressions P_{-i} and P_{+i} in case of the described tableau system for **TL**, for any $i, j \in \mathbb{N}$ and any name letter $P \in \mathsf{Ln}$. The auxiliary expressions also feature indices.

Such expressions will be specified by means of set TeA, subset of Cartesian product $\{\alpha_1, \alpha_2, \alpha_3, \dots\} \times \{\beta_1, \beta_2, \beta_3, \dots\}$, where for any $i, j \in \mathbb{N}$, if $\langle \alpha_i, \beta_j, \rangle \in \mathsf{TeA}$,

4 In Chapter Two, we adopted the simplest variant. We could have, however, adopted the set of tableau expressions for **CPL** different from the set of formulas.

then β_j is an ordered n-tuple $\langle k_1, k_2, \dots, k_n \rangle$ of indices present in that sequence in expression α_i, for some $n, k_1, k_2, \dots, k_n \in \mathbb{N}$.

Set $\mathsf{TeA} \subseteq \mathsf{Te}$, moreover TeA may be empty, because the construction of tableau proof for a given logic may not require at all a richer set than what is required by definition 5.15, or its definition uses a different set of auxiliary expressions than TeA.

Definition 5.22 (Function selecting indices)**.** *Function selecting indices* will be called function $* : P(\mathsf{Te}) \longrightarrow P(\mathbb{N})$ defined for any $i, j \in \mathbb{N}$ with conditions:

- for any formula $A \in \mathsf{For}$, $*(\{A^i\}) = \{i\}$
- for any $\langle \alpha_i, \beta_j \rangle \in \mathsf{TeA}$, $*(\{\langle \alpha_i, \beta_j \rangle\}) = \{k : k$ is an element of $\beta_j\}$
- for any $X \subseteq \mathsf{Te}$, if $|X| > 1$, then $*(X) = \bigcup\{*(\{y\}) : y \in X\}$.

Therefore, for any subset of set $\{A^i : A \in \mathsf{For}, i \in \mathbb{N}\} \subseteq \mathsf{Te}$ function $*$ selects all indices present in the expressions from that set.

We will now proceed to the general conditions that specify the relation of similarity between the sets of expressions, regardless of whether or not set TeA is non-empty and set Te contains any other specific auxiliary expressions. Therefore, in the definition of similarity we assume the condition of having the same cardinality.

Definition 5.23 (Similar set of expressions)**.** Let X, $Y \subseteq \mathsf{Te}$. We shall state that X is *similar* to Y iff:

- X is t-consistent iff Y is t-consistent
- X and Y have the same cardinality
- there exists such bijection $h : *(X) \longrightarrow *(Y)$ that:
  - for any formula $A \in \mathsf{For}$ and $i \in \mathbb{N}$, $A^i \in X$ iff $A^{h(i)} \in Y$.
  - for any $i, n, k_1, k_2, \dots, k_n \in \mathbb{N}$, there exist such $j \in \mathbb{N}$, $\langle \alpha_i, \beta_j \rangle \in \mathsf{TeA}$ that $\langle \alpha_i, \beta_j \rangle \in X$ and $\beta_j = \langle k_1, k_2, \dots, k_n \rangle$ iff there exist such $l \in \mathbb{N}$, $\langle \alpha_i, \beta_l \rangle \in \mathsf{TeA}$ that $\langle \alpha_i, \beta_l \rangle \in Y$ and $\beta_l = \langle h(k_1), h(k_2), \dots, h(k_n) \rangle$.

From definition of similarity 5.23 the following conclusion results.

Corollary 5.24. *Let $X, Y \subseteq \mathsf{Te}$. If X is similar to Y, then Y is similar to X.*

For further work, we need a definition of relation that occurs between interpretation of formulas $\mathfrak{I}$ and subset of tableau expressions $\{A^i : A \in \mathsf{For}, i \in \mathbb{N}\} \subseteq \mathsf{Te}$.

Definition 5.25. Let $i \in \mathbb{N}$. By $\Vdash_i$ we mean any such relation specified on Cartesian product $\mathbf{I} \times \mathsf{For}$ that for any formula $A \in \mathsf{For}$ and any interpretation $\mathfrak{I} \in \mathbf{I}$:

- if $\mathfrak{I} \models A$, then $\mathfrak{I} \Vdash_i A$.
- $\mathfrak{I} \Vdash_i A$ iff it is not the case that $\mathfrak{I} \Vdash_i f(A)$ (for short: $\mathfrak{I} \nVdash_i A$).
- if for certain $j \in \mathbb{N}$, $\mathfrak{I} \Vdash_j A$, then $\mathfrak{I} \Vdash_i A$.

By $\Vdash$ we will mean set $\{\langle \mathfrak{I}, A\rangle : \mathfrak{I} \in \mathbf{I}, A \in \mathsf{For}$ and for some $i \in \mathbb{N}$ $\mathfrak{I} \Vdash_i A\}$.

Remark 5.26. For our considerations, we adopt any, but fixed relation $\Vdash$. It will remain unchanged until the end of this chapter.

Remark 5.27. Intuitively, relation $\Vdash$ is in a sense expansion of relation $\models$ onto set of tableau expressions $\{A^i : A \in \mathsf{For}, i \in \mathbb{N}\} \subseteq \mathsf{Te}$. We used "in a sense", because in the cases of **CPL** and **TL** we can identify it with relation $\models$, as for each formula A, of one of these logics, $A^i = A$, for any $i \in \mathbb{N}$. On the other hand, in the event of modal logic (e.g. **S5**), notation $\mathfrak{I} \Vdash_i A$ may indicate that if based on model $\mathfrak{M}$, that we identify with interpretation $\mathfrak{I}$, we construct model $\mathfrak{M}'$, where the distinguished world will be corresponded by index i, then $\mathfrak{M}' \models A$.

We will now specify the general conditions describing the interpretation appropriate for set of expressions.

Definition 5.28 (Interpretation appropriate for set of expressions)**.** Let $\mathfrak{I} \in \mathbf{I}$ be an interpretation, and let $X \subseteq \mathsf{Te}$ be a set of tableau expressions. We shall state that interpretation $\mathfrak{I}$ is *appropriate* for set X iff:

1. X is t-consistent
2. for any $i \in \mathbb{N}$ and any $A \in \mathsf{For}$, if $A^i \in X$, then $\mathfrak{I} \Vdash_i A$.

5.4 Tableau rules

Now, we will proceed to the conditions that should be satisfied by the tableau rules. So, let us define the general concept of rule.

Definition 5.29 (Rule)**.** Assume that $P(\mathsf{Te})$ is a set of all subsets of set Te. Let $P(\mathsf{Te})^n$ be n-element of Cartesian product $\underbrace{P(\mathsf{Te}) \times \cdots \times P(\mathsf{Te})}_{n}$, for some $n \in \mathbb{N}$, whereas $\bigcup_{n \in \mathbb{N}} P(\mathsf{Te})^n$ be a union of all n-ary products such that $n \geq 2$.

- *Rule* will be called any subset $R \subseteq \bigcup_{n \in \mathbb{N}} P(\mathsf{Te})^n$ such that if $\langle X_1, \ldots, X_n\rangle \in R$, then the following conditions are satisfied:
  - $X_1 \subset X_i$, for any $1 < i \leq n$
  - X_1 is a t-consistent set
  - if $k \neq l$, then $X_k \neq X_l$, for any $1 < k, l \leq n$
  - (Closure under similarity) for any set of expressions Y_1 such that Y_1 is similar to X_1, there exist such sets of expressions $Y_2, \ldots, Y_n$ that $\langle Y_1, \ldots, Y_n\rangle \in R$ and for any $1 < i \leq n$, Y_i is similar to X_i

- (Existence of core of rule) for some finite set $Y \subseteq X_1$ there exists ordered n-tuple $\langle Z_1, \ldots, Z_n \rangle \in R$ such that:
  1. $Z_1 = Y$
  2. for any $1 < i \leq n$, $Z_i = Z_1 \cup (X_i \setminus X_1)$
  3. there does not exist proper subset $U_1 \subset Y$ and n-tuple $\langle U_1, \ldots, U_n \rangle \in R$
- (Closure under expansion) for any t-consistent set of expressions Z_1, such that $X_1 \subset Z_1$ and for each $1 < i \leq n$, X_i is not similar to any subset Z_1 containing X_1, there exist such set of expressions $Z_2, \ldots, Z_n$ that $\langle Z_1, \ldots, Z_n \rangle \in R$ and for any $1 < i \leq n$, X_i is similar to $X_1 \cup (Z_i \setminus Z_1)$
- (Closure under finite sets) if X_1 is a finite set, then for each $1 < i \leq n$, X_i is a finite set
- (Closure under subsets) for any subset $X' \subseteq X_1$, if for certain set $Y \subseteq X_1$ there exists such n-tuple $\langle W_1, \ldots, W_n \rangle \in R$ that:
  1. $W_1 = Y$
  2. for any $1 < i \leq n$, $W_i = W_1 \cup (X_i \setminus X_1)$
  3. $W_1 \subseteq X'$

  then there exist such sets of expressions $Z_1, \ldots, Z_n$ that:
  1. $\langle Z_1, \ldots, Z_n \rangle \in R$
  2. $Z_1 = X'$
  3. for any $1 < i \leq n$, $Z_i = Z_1 \cup (X_i \setminus X_1)$.

- We shall state that rule R has been *applied* to set X_1 iff for certain $1 < i \leq n$, exactly one set X_i was selected from certain n-tuple $\langle X_1, \ldots, X_n \rangle \in R$.

Remark 5.30. Note that the general concept of rule we have introduced with definition 5.29, is in a way less general than the general concepts of rules used in the previous chapters — since we have added several additional conditions that were not present before. For we will not use a specific set of rules in further proofs, but we will define another conditions that should be jointly met by the set of tableau rules selected for axiomatization of the tableau system.

However, in some respects, definition 5.29 is more general than the definitions of rules in the previous chapters. In those cases, the rule was to be a subset of Cartesian product $R \subseteq P(\mathsf{Te})^n$, where $n \geq 2$. And here, the rule is defined as a subset of the union under Cartesian products $R \subseteq \bigcup_{n \in \mathbb{N}} P(\mathsf{Te})^n$, where $n \geq 2$. Of course, the rules that meet the first condition also meet the second one. The more general condition was provided for a number of reasons.

First of all, even in the case of structural definition of tableau rules — as we have done so far — there may be a situation in which ordered pairs of different numbers of elements belong to a single rule. An example is a logic with relating connectives, in which we sometimes have to describe relationships between propositional letters when distributing expressions in tableaux. In this case, the number of elements in a given n-tuple may depend on the number of propositional letters in a given formula.[5]

Secondly, even if we define the tableau rules structurally, it is also possible to define them for some system in a more complex way. We could, for example, sum up some (or even all) of the rules described in the chapter on the tableau system for **CPL**, to get for instance rule $R' = R_{\neg\neg} \cup R_{\vee}$. That rule would consist of two types of n-tuples: ordered pairs and ordered triples.

Finally, we can build tableau systems for non-classical reasoning (inferences), in which we draw conclusions in a non-deductive way. In such systems, there may occur a need for unstructured definition of tableau rules, e.g. in such a way that from a given structurally described set of premises it is possible to move in the tableau to one or more branches, depending on which subformulas the premises are composed of. Excluding certain conditions from definition 5.29, that definition may be useful in such cases.

We will now frame a general definition of the core of rule in a given set.

Definition 5.31 (Core of rule)**.** Let R be a rule and $n \in \mathbb{N}$. Let $\langle X_1, \ldots, X_n \rangle \in R$ and $\langle Z_1, \ldots, Z_n \rangle \in R$. We shall state that $\langle Z_1, \ldots, Z_n \rangle \in R$ is a *core of rule R in set* $\langle X_1, \ldots, X_n \rangle$ iff

1. $Z_1 \subseteq X_1$
2. there do not exist proper subset $U_1 \subset Z_1$ and n-tuple $\langle U_1, \ldots, U_n \rangle$ such that $\langle U_1, \ldots, U_n \rangle \in R$ and $U_i \setminus U_1 = Z_i \setminus Z_1$, for any $1 < i \leq n$
3. for any $1 < i \leq n$, $Z_i = Z_1 \cup (X_i \setminus X_1)$.

By definition of rule 5.29 (Existence of core of rule), we get the following conclusion.

5 Although the book does not directly consider logics with relating connectives, the results of this chapter also apply to the tableau systems for such logics. An introduction to relating logics was presented inter alia in [12], [7] and [11]. The simplest example of a tableau system of logic with relating connectives is described in paper [10]. In turn, the issue of rules containing various n-tuples appears in those logics with relating connectives for which the semantic structure is constrained by various conditions motivated by philosophically oriented interpretation of connectives. Such an approach to tableau methods for logics with relating connectives in the context of causality can be found in [11] as well as connexivity in [12], [13]. It is worth noting that in the case of [13] in a sense relating logics were combined with modal ones and two tableau approaches were joined.

Corollary 5.32. *Let R be a rule and $n \in \mathbb{N}$. If n-tuple $\langle X_1, \ldots, X_n \rangle \in R$, then there exists n-tuple $\langle Y_1, \ldots, Y_n \rangle \in R$ such that $\langle Y_1, \ldots, Y_n \rangle$ is the core of rule R in set $\langle X_1, \ldots, X_n \rangle$.*

We will now define two additional technical concepts that will allow us to frame a general definition of a set of tableau rules.

Let $X \subseteq \mathsf{Te}$ be a set of tableau expressions and let $\mathbf{R}$ be a set of rules. By $\mathbf{R}_X$ we will mean a set of all and only such rules from set $\mathbf{R}$ that are applicable to set X. Formally, $R \in \mathbf{R}_X$ iff $R \in \mathbf{R}$ and there exists such n-tuple $\langle Y_1, \ldots, Y_n \rangle \in R$ that $Y_1 = X$.

Let $R \in \mathbf{R}_X$, by R_X we will mean a set of all and only such n-tuples from R that their first element equals X, and if some of the remaining elements of two n-tuples that belong to R_X differ, then these two n-tuples have different input sets of the core of rule. Formally, for any $n \in \mathbb{N}$, $\langle Y_1, \ldots, Y_n \rangle \in R_X$ iff:

- $\langle Y_1, \ldots, Y_n \rangle \in R$ and $Y_1 = X$
- for any set of expressions $Z_1, \ldots, Z_n, Z'_1, \ldots, Z'_n, Y'_1, \ldots, Y'_2 \subseteq \mathsf{Te}$, if:
  - $\langle Z_1, \ldots, Z_n \rangle \in R_X$
  - $\langle Y_1, \ldots, Y_n \rangle \neq \langle Z_1, \ldots, Z_n \rangle$
  - $\langle Y'_1, \ldots, Y'_2 \rangle$ is the core of rule R in set $\langle Y_1, \ldots, Y_n \rangle$
  - $\langle Z'_1, \ldots, Z'_n \rangle$ is the core of rule R in set $\langle Z_1, \ldots, Z_n \rangle$

  then $Y'_1 \neq Z'_1$.

The limitations in the definition of set R_X are the results of the fact that some rules, e.g. $R_\Diamond$ in the described tableau system for **S5**, may introduce completely new expressions that are absent in any form in the previous portions of the proof.

In the case of rule $R_\Diamond$, we can introduce to the branch some set $\{\langle A, i \rangle, irj\}$, where $A \in \mathsf{For}_{\mathbf{S5}}$ and $i, j \in \mathbb{N}$, selected from among many such sets. Usually, there exist multiple such n-tuples that belong to $R_\Diamond$ which are applicable to set X, if $\langle \Diamond A, i \rangle \in X$, after even one application we cannot apply rule $R_\Diamond$ to set X anymore due to expression $\langle \Diamond A, i \rangle \in X$, because of the limitations in the application of this rule (see example 4.24). So, we want set R_X to include only one such n-tuple, since we can only use one.

With the above concepts, we can proceed to the definition of set of tableau rules.

Definition 5.33 (Tableau rules)**.** Let $\mathbf{R}$ be a set of rules. We shall state that $\mathbf{R}$ is a *set of tableau rules* iff

1. $\mathbf{R}$ is a finite set
2. for any $X \subseteq \mathsf{Te}$, if X is a finite set, then for any rule $R \in \mathbf{R}_X$, each set R_X is a finite set.

So, set of tableau rules **R** must include a finite number of rules, and what is more, for any finite set of expressions, each of rules R that belong to **R** can be applied a finite number of times — taking account of the set of n-tuples that belong to R_X.

Remark 5.34. We adopt any, but fixed set of tableau rules **R**. That set will remain unchanged until the end of this chapter. It is worth noting again that all further tableau concepts: branches and tableaux of different types will depend on set **R**.

5.4.1 Branches

Conventionally, another concept in our theory that will be discussed is the concept of branch. It is a concept that depends on the notion of the tableau rule because branches are created by applying rules. Branches — as mentioned before — are setwise objects consisting of sets. The below definition corresponds to all the definitions of branch used so far, only that it depends on the general notion of set of tableau rules **R**.

Definition 5.35 (Branch)**.** Let $K = \mathbb{N}$ or $K = \{1,2,\ldots,n\}$, where $n \in \mathbb{N}$. Let X be any set of expressions. *A branch* (or *a branch beginning with* X) will be called any sequence $\phi : K \longrightarrow P(\mathsf{Te})$ that meets the following conditions:

1. $\phi(1) = X$
2. for any $i \in K$: if $i+1 \in K$, then there exists such rule $R \in \mathbf{R}$ and such n-tuple $\langle Y_1,\ldots Y_n\rangle \in R$ that $\phi(i) = Y_1$ and $\phi(i+1) = Y_k$, for some $1 < k \leq n$.

Having two branches ϕ, ψ such that $\phi \subset \psi$ we shall state that:

- ϕ is a sub-branch of ψ
- ψ is a super-branch of ϕ.

Denotation 5.36. From now on — when speaking of the branches constructed by the application of rules from set **R** — for the sake of convenience, we will use the following notations or denotations:

1. $X_1,\ldots,X_n$, where $n \geq 1$
2. $\langle X_1,\ldots,X_n\rangle$, where $n \geq 1$
3. abbreviations: ϕ_M (where M is a domain ϕ, i.e. $\phi : M \longrightarrow P(\mathsf{Te})$)
4. or — to denote branches — small Greek letters: ϕ, ψ, etc.

The sets of branches, in turn, we shall denote with capital Greek letters: Φ, Ψ, etc. Furthermore, the domain cardinality of a given branch K we shall sometimes call a *length* of that branch.

All the so far considered branches have been specific cases of the above concept, assuming an appropriate set of rules **R**.

Let us now introduce the general definition of addition of branches.

Definition 5.37 (Addition of branches)**.** Let $\phi : \{1, \ldots, n\} \longrightarrow P(\mathsf{Te})$ and $\psi : M \longrightarrow P(\mathsf{Te})$ be branches, for some $n \in \mathbb{N}$ and $M \subseteq \mathbb{N}$, and let $\phi(n) = \psi(1)$. The results of the operation $\phi \oplus \psi$ is function $\varphi : K \longrightarrow P(\mathsf{Te})$ defined as follows:

1. if $M = \mathbb{N}$, then $K = \mathbb{N}$
2. if $|M| \in \mathbb{N}$, then $K = \{1, \ldots, n, n+1, n+2, \ldots, n+|M|-1\}$
3. for each $i \in K$
   a. if $1 \leq i \leq n$, $\varphi(i) = \phi(i)$
   b. if $i > n$, then $\varphi(i) = \psi((i-n)+1)$.

From definition of branch 5.35 and definition of addition of branches 5.37, follows an analogous conclusion as in the case of the tableau system for logic **S5**.

Corollary 5.38. *Let $\phi : \{1, \ldots, n\} \longrightarrow P(\mathsf{Te})$ and $\psi : M \longrightarrow P(\mathsf{Te})$ be branches, for some $n \in \mathbb{N}$ and $M \subseteq \mathbb{N}$, and let $\phi(n) = \psi(1)$. Then $\phi \oplus \psi$ is also a branch.*

5.4.2 Closed and open branches

An important classification of branches is the division into closed and open branches. A branch is closed when, applying the rules in subsequent steps, we have reached a t-inconsistent set. Below, we present the definition which is directly based on set of tableau rules **R**, as it refers to definition of branch 5.35.

Definition 5.39 (Closed/open branch)**.** Branch $\phi : K \longrightarrow P(\mathsf{Te})$ will be called *closed* iff $\phi(i)$ is a t-inconsistent set for some $i \in K$. A branch will be called *open* iff it is not closed.

From the above definition 5.39, definition of tableau rules 5.33 and definition of branch 5.35, the following conclusion results.

Corollary 5.40. *If branch $\phi : K \longrightarrow P(\mathsf{Te})$ is closed, then $|K| \in \mathbb{N}$.*

Again, in the case of a closed branch, the t-inconsistent sequence element is the last element and no rule can be applied to it anymore to extend the branch. For the tableau rules have been defined in such a way that they cannot be applied to t-inconsistent sets.

5.4.3 Maximal branches

One more important concept in the construction of a tableau system is the concept of a maximal branch. The definition of maximal branch is based on the concept of strong similarity. As we already know from the previous chapter, the concept of strong similarity of sets of expressions is a special case of the similarity of sets. Below, we provide its version that is generalized to the context of rules from set **R**.

Definition 5.41 (Strong similarity)**.** Let rule $R \in \mathbf{R}$ and let $\langle X_1, \ldots, X_n \rangle \in R$, for some $n \in \mathbb{N}$. On any set of expressions $W \subseteq \mathsf{Te}$ we will state that it is *strongly similar to* set X_i, where $1 < i \leq n$, iff

1. W is similar to X_i
2. for certain n-tuple $\langle Y_1, \ldots, Y_n \rangle$, which is the core of rule R in set $\langle X_1, \ldots, X_n \rangle$, the following conditions are satisfied:
   a. for certain $W' \subseteq W$, $Y_1 \subseteq W'$
   b. W' is similar to $Y_1 \cup (X_i \setminus X_1)$.

Having adopted the concept of strong similarity, we can proceed to the concept of maximal branch in the general version, also referred to set of tableau rules **R**.

Definition 5.42 (Maximal branch)**.** Let $\phi : K \longrightarrow P(\mathsf{Te})$ be a branch. We shall state that ϕ is *maximal* iff it meets one of the below conditions:

1. ϕ is closed
2. for any rule $R \in \mathbf{R}$, any $n \in \mathbb{N}$ and any n-tuple $\langle X_1, \ldots, X_n \rangle \in R$, if $\phi(k) = X_1$, for certain $k \in K$, then for some $j \in K$, there exist $\phi(j)$ and such set of expressions $W \subseteq \mathsf{Te}$ that for some $1 < i \leq n$, W is strongly similar to X_i and $W \subseteq \phi(j)$.

Remark 5.43. We will repeat here the remark from the previous chapter. According to the above definition, a maximal branch is closed or, in a sense, closed under effect of rules (both conditions do not necessarily have to be mutually exclusive). Closure under rules means that if a branch is not closed and it was possible to apply some rule to one of its elements, then some of the branch elements includes a set strongly similar to the one that could have been a result of application of that rule. Set W is to be contained in one of the elements of branch $\phi(j)$, and not necessarily be identical to it, since we applied the rule to set $\phi(j-1)$, which can be a proper superset of set X, and consequently we obtained more expressions, and what is more, those elements could have been obtained as a result of application of another rule.

Maximal branches as defined by the definition 5.42 can be either finite or infinite. There occurs an analogous case here as in the considerations on logic **S5**: if the branch is finite and there does not exist super-branch, then it is also a maximal branch.

Corollary 5.44. *If branch ϕ is finite in length, and there does not exist such branch ψ that $\phi \subset \psi$, then ϕ is a maximal branch.*

Proof. Take any branch ϕ which is finite and there does not exist such branch ψ that $\phi \subset \psi$. Now, assume that ϕ is not closed. If it does not meet the second conditions of definition 5.44, then since ϕ is finite, so there exists branch ψ such that $\phi \subset \psi$, which obviously contradicts the assumption. □

So we have a general definition of maximal branch, which includes both finite cases — especially in systems that feature the finite branch property, and infinite cases, such as those that occur, for example, in modal logics.

The described concepts, therefore, apply to systems in which, by building a tableau proof using tableau tools and looking for maximal branches, we may be dealing with infinite branches. This is the case when a branch cannot be closed or certain rule application sequences are repeated.

From definitions 5.39 and 5.42 we get the following conclusion again.

Corollary 5.45. *Each closed branch is maximal.*

5.4.4 Branch consequence relation

We will now move on to the general concept of branch consequence which we will define using the following concepts: branch, maximal branch, closed branch, and denotation 5.19.

Definition 5.46 (Branch consequence)**.** Let $X \subseteq \mathsf{For}$ and $A \in \mathsf{For}$. We shall state that A is *branch consequence of* X (for short: $X \triangleright A$) iff there exists such finite set $Y \subseteq X$ and such index $i \in \mathbb{N}$ that each maximal branch beginning with set $Y^i \cup \{f(A)^i\}$ is closed. By $X \not\triangleright A$, we mean that A is not branch consequence X.

The general concept of branch consequence relation corresponds to the so far defined concepts of branch consequence relation, taking account of remark 5.18. This remark is valid for all tableau concepts defined hereafter. So, when constructing a branch or tableau for some set of formulas $X \cup \{A\}$, we begin with set $X^i \cup \{f(A)^i\}$, for some index $i \in \mathbb{N}$.

5.5 Tableaux

In this subchapter, we will move on to the general definition of tableau and various variants of tableaux. However, we will start with an auxiliary concept of maximality in set of branches which we already have used in the previous chapters.

Definition 5.47 (Maximal branch in the set of branches)**.** Let Φ be a set of branches and let branch $\psi \in \Phi$. We shall state that ψ is *maximal in set* Φ (for short: Φ–*maximal*) iff there is no such branch $\phi \in \Phi$ that $\psi \subset \phi$.

We can now move on to the general concept of tableau.

Definition 5.48 (Tableau)**.** Let $X \subseteq \mathsf{For}$, $A \in \mathsf{For}$ a Φ be a set of branches. Ordered triple $\langle X, A, \Phi \rangle$ will be called *a tableau for* $\langle X, A \rangle$ (or for short: *tableau*) iff the below conditions are met:

1. Φ is a non-empty subset of set of branches beginning with set $X^i \cup \{f(A)^i\}$, for some index $i \in \mathbb{N}$ (i.e. if $\psi \in \Phi$, then $\psi(1) = X^i \cup \{f(A)^i\}$)
2. each branch contained in Φ is Φ-maximal
3. for any $n, i \in \mathbb{N}$ and any branches $\psi_1, \ldots, \psi_n \in \Phi$, if:
   - i and $i+1$ belong to domains of functions $\psi_1, \ldots, \psi_n$
   - for any $1 < k \leq n$ and any $o \leq i$, $\psi_1(o) = \psi_k(o)$

   then there exists such rule $R \in \mathbf{R}$ and such ordered m-tuple $\langle Y_1, \ldots, Y_m \rangle \in R$, where $1 < m$, that for any $1 \leq k \leq n$:
   - $\psi_k(i) = Y_1$
   - and there exists such $1 < l \leq m$ that $\psi_k(i+1) = Y_l$.

The above concept of tableau covers all notions of tableau considered so far in a book, with a properly defined set of tableau rules **R**.

When considering the tableaux in general, we can also generalize the concept of redundant branch which is useful for the definition of complete tableau.

Definition 5.49 (Redundant variant of branch)**.** Let ϕ and ψ be such branches that for some numbers i and $i+1$ that belong to their domains, it is the case that for any $j \leq i$, $\phi(j) = \psi(j)$, but $\phi(i+1) \neq \psi(i+1)$. We shall state that branch ψ is a *redundant variant* of branch ϕ iff:

- there exists such rule $R \in \mathbf{R}$ and such n-tuple $\langle X_1, \ldots, X_n \rangle \in R$ that $\phi(i) = X_1$ and $\phi(i+1) = X_j$, for certain $1 < j \leq n$
- there exists rule $R \in \mathbf{R}$ and such m-tuple $\langle Y_1, \ldots, Y_m \rangle \in R$, where $m > n$, that $X_1 = \psi(i) = Y_1$ and:
  1. $\psi(i+1) = Y_k$, for certain $1 < k \leq m$
  2. for any $1 < l \leq n$ there exists such $1 < o \leq m$ that $o \neq k$ and $X_l = Y_o$.

Let Φ, Ψ be sets of branches and $\Phi \subset \Psi$. We shall state that Ψ is a *redundant superset* Φ iff for any branch $\psi \in \Psi \setminus \Phi$ there exists such branch $\phi \in \Phi$ that ψ is a redundant variant of ϕ.

Making use of the general concept of redundant superset of branches we can define the general concept of complete tableau.

Definition 5.50 (Complete tableau)**.** Let $\langle X, A, \Phi \rangle$ be a tableau. We shall state that $\langle X, A, \Phi \rangle$ is *complete* iff:

1. each branch contained in Φ is maximal
2. any set of branches Ψ such that:
   a. $\Phi \subset \Psi$
   b. $\langle X, A, \Psi \rangle$ is a tableau

   is a redundant superset of Φ.

A tableau is *incomplete* iff the tableau is not complete.

Now we can define the general concept of closed and open tableaux.

Definition 5.51 (Closed/open tableau)**.** Let $\langle X, A, \Phi \rangle$ be a tableau. We shall state that $\langle X, A, \Phi \rangle$ is *closed* iff the below conditions are met:

1. $\langle X, A, \Phi \rangle$ is a complete tableau
2. each branch contained in Φ is closed.

A tableau is *open* iff the tableau is not closed.

By virtue of the above definitions of closed tableau and complete tableau, we get a conclusion which makes up a generalization of the analogous conclusions from the preceding chapters.

Corollary 5.52. *Each closed tableau is a complete tableau.*

5.6 Completeness theorem

In this chapter, we will define several general concepts and establish facts that will allow us to prove a claim from which we can deduce the theorem on completeness for the tableau system that meets the conditions given below.

In the first place, we will address the concepts used to demonstrate the relationship between relation $\vDash$ and $\rhd$. We will begin with the definition of branch generating interpretation.

Definition 5.53 (Branch generating interpretation)**.** Let Φ be a set of all open and maximal branches that contain some tableau equivalents of formulas and let $\mathfrak{I}$ be an interpretation of formulas. We shall state that branch $\phi \in \Phi$ *generates interpretation* $\mathfrak{I}$ iff there exists such function $\succcurlyeq: \Phi \longrightarrow \mathbf{I}$ that $\succcurlyeq(\phi) = \mathfrak{I}$.

Remark 5.54. The general definition of generating interpretation of formulas by branch is purely auxiliary and redundant in nature. For it is difficult to generally establish a definition of function $\succcurlyeq$. However, in the case of specific logics or whole classes of logics, this concept takes on a very specific meaning — we can then describe the transition from the open and maximal branch to the construction of interpretation. We will present this issue in the next chapter, describing examples of application.

Another important concept is the concept of set of interpretations good for rules.

Definition 5.55 (Interpretations good for rules)**.** Let

- **R** be a set of tableau rules
- ϕ be an open and maximal branch
- $X^i \subseteq \bigcup \phi$, for some non-empty $X \subseteq \mathsf{For}$ and some $i \in \mathbb{N}$
- **I** be a set of interpretations of formulas.

We shall state that set **I** is *good* for set of rules **R** iff branch ϕ generates such interpretation $\mathfrak{I}$ that:

- $\mathfrak{I} \in \mathbf{I}$
- $\mathfrak{I} \models X$.

We will now define the general concept of closure under rules.

Definition 5.56 (Closure under rules)**.** Let $X \subseteq \mathsf{Te}$. We shall state that $Y \subseteq \mathsf{Te}$ is a *closure* of set X under rules **R** iff Y is a set that meets the following conditions:

- $X \subseteq Y$
- for any rule $R \in \mathbf{R}$ and any n-tuple $\langle Z_1, Z_2, \ldots, Z_n \rangle \in R$, where $n \in \mathbb{N}$, if $X \subseteq Z_1 \subseteq Y$, then $Z_j \subseteq Y$, for some $2 \leq j \leq n$.

On set Y we will also state that is a *closure*.

For any set of expressions, there exists at least one closure, at times there may exist more closures.

Using the above concept of closure, we can move on to the verbalization and proof of the following lemma.

Lemma 5.57 (On the existence of open and maximal branch)**.** *Let $X \subseteq \mathsf{For}$ and $i \in \mathbb{N}$. If for each finite $Y \subseteq X$, there exists a maximal and open branch beginning with Y^i, then there exists a closure of set X^i under rules* **R** *which is an open and maximal branch.*

Proof. Take any $X \subseteq \mathsf{For}$, $i \in \mathbb{N}$, and assume that $(*)$ for each finite $Y \subseteq X$ there exists an open and maximal branch beginning with set Y^i.

Next, we specify the set of all maximal and open branches that begin with set Y^i, for some finite $Y \subseteq X$ — we will denote that set as **X**.

We define set $\overline{\mathbf{X}}$, through the following conditions:

1. $\overline{\mathbf{X}} \subseteq \mathbf{X}$
2. for any two branches ϕ and ψ contained in $\mathbf{X}$, if there exist such $i, k \in \mathbb{N}$ that $\phi(i) \cup \psi(k)$ is a t-inconsistent set, then $\phi \notin \overline{\mathbf{X}}$ or $\psi \notin \overline{\mathbf{X}}$
3. $\overline{\mathbf{X}}$ is a maximal set among those subsets $\mathbf{X}$ that meet conditions 1 and 2.

There exists at least one set $\overline{\mathbf{X}}$ such that $\overline{\mathbf{X}} \subseteq \mathbf{X}$. We take one of such sets $\overline{\mathbf{X}}$ and denote it as $\overline{X}$.

Consider set $\bigcup\{\phi(1) : \phi \in \overline{X}\}$. Note that $(**)$ $X^i \subseteq \bigcup\{\phi(1) : \phi \in \overline{X}\}$. For when $X^i \nsubseteq \bigcup\{\phi(1) : \phi \in \overline{X}\}$, there would exist such $x \in X^i$ that $x \notin \bigcup\{\phi(1) : \phi \in \overline{X}\}$ and, consequently, for any such branch $\psi \in \mathbf{X}$ that $x \in \psi(1)$, $\psi(1) \subseteq X^i$ and $\psi(1)$ is a finite set, it would be the case that $\psi \notin \overline{X}$. Then, however, for some finite set $Y^i \subseteq X^i$ there would exist no maximal and open branch beginning with set $Y^i \cup \{x\}$ which would contradict assumption $(*)$.

We define the condition that specifies new set $\overline{\overline{X}}$:
$U \in \overline{\overline{X}}$ iff there exists such branch ϕ that $\phi \in \overline{X}$ and $U = \bigcup \phi$
Now, we can define set $Z = \bigcup \overline{\overline{X}}$.

We claim that Z is a closure of set X^i under tableau rules $\mathbf{R}$ (definition 5.56), and that Z is an open and maximal branch.

First, we will show that Z is a closure of set X^i, thus that it meets conditions of definition of closure 5.56.

Note that $X^i \subseteq Z$, since $(**)$ $X^i \subseteq \bigcup\{\phi(1) : \phi \in \overline{X}\}$, and by definition of set Z, $\bigcup\{\phi(1) : \phi \in \overline{X}\} \subseteq Z$.

Now, take any rule $R \in \mathbf{R}$ and any n-tuple $\langle U_1, \ldots, U_n \rangle \in R$, for some $n \in \mathbb{N}$, and assume that $X^i \subseteq U_1 \subseteq Z$. From definition 5.33, it follows that there exists such n-tuple $\langle U'_1, \ldots, U'_n \rangle \in R$ that:

- for each $1 \leq j \leq n$, U'_j is such a minimal and finite set that if U_j is not such a minimal and finite set such that $\langle U_1, \ldots, U_n \rangle \in R$, then $U'_j \subset U_j$
- for any $1 < j \leq n$, $U_j \setminus U_1 = U'_j \setminus U'_1$.

Consequently, assuming that $U'_1 \subseteq Z$, we must show that for certain $1 < l \leq n$, $U'_l \subseteq Z$, since $U'_l \cup U_1 = U_l$. Since $U'_1 \subseteq Z$ and U'_1 is a finite set, thus there exists a finite number of such branches $\phi_1, \phi_2, \ldots, \phi_o$ in set $\overline{X}$ that for certain $k \in \mathbb{N}$, $U'_1 \subseteq \phi_1(k) \cup \phi_2(k) \cup \cdots \cup \phi_o(k)$. So, set $\overline{X}$ contains such branch ψ that $\psi(1) = \phi_1(1) \cup \phi_2(1) \cup \cdots \cup \phi_o(1)$ and $U'_1 \subseteq \psi(m)$, for certain $m \in \mathbb{N}$, and since $\phi_1(k) \cup \phi_2(k) \cup \cdots \cup \phi_o(k)$ is a t-consistent set, so set $\overline{X}$ contains such maximal branch ψ' that — by definition 5.42 — for certain $1 < l \leq n$, $U'_l \subseteq \bigcup \psi'$. Finally, $U'_l \subseteq Z$, since by construction Z, $\bigcup \psi \subseteq Z$.

We will now move on to showing that Z is an open and maximal branch.

From the definition of branch — 5.35 — it follows that Z is a branch.

Whereas by construction of Z, Z is an open branch, i.e. no subset Z is t-inconsistent, by definition $\overline{X}$.

Let us now check if Z is a maximal branch. According to definition 5.42, we assume that there exists such rule $R \in \mathbf{R}$ and such n-tuple $\langle X_1, \ldots, X_n \rangle \in R$, for some $n \in \mathbb{N}$ that $X_1 = Z$. By definition of tableau rules 5.33 (Existence if core) and (Closure under subsets), there exists such n-tuple $\langle X_1' \ldots, X_n' \rangle \in R$ that for any $1 < j \leq n$, $X_j \setminus X_1 = X_j' \setminus X_1'$ and $X^i \subseteq X_1' \subseteq Z$. Since Z is a closure X^i, so $X_j' \subseteq Z$, for certain $1 < j \leq n$, by definition 5.56. Therefore, $X_j \subseteq Z$, since $X_j = X_1 \cup X_j'$. But then $X_1 \nsubseteq X_j$, which by definition 5.33 is out of the question. Consequently, there exists no such tableau rule R and n-tuple $\langle X_1, \ldots, X_n \rangle \in R$ that $X_1 = Z$, for some $n \in \mathbb{N}$. Therefore, Z is a maximal branch, by definition 5.42. □

Now, we will verbalize and prove a fact that is needed to demonstrate the relationship between relation $\triangleright$ and the existence of a closed tableau.

Proposition 5.58. *Let $X \subseteq \mathsf{Te}$. If X is a finite set, then there exists such maximal branch ϕ that $\phi(1) = X$.*

Proof. Take any subset $X \subseteq \mathsf{Te}$. The set of all branches beginning with set of expressions X will be denoted as $\mathbf{X}$. Set $\mathbf{X}$ is non-empty as by definition of branch 5.35, such mapping $\psi : \{1\} \longrightarrow P(\mathsf{Te})$ that $\psi(1) = X$, is a branch.

We have two options: (1) there exists a closed branch in set $\mathbf{X}$, or (2) there does not exist any closed branch in set $\mathbf{X}$.

If case (1) occurs, then by definition of maximal branch 5.42, there exists such maximal branch ϕ that $\phi(1) = X$.

Assume that case (1) does not occur.

Let $Y \subseteq \mathsf{Te}$ be any finite set of expressions. By definition 5.33, the number of tableau rules that belong to set $\mathbf{R}_Y$ is finite and different from zero. Now, assume there is j of them, for some $j \in \mathbb{N}$. We assign to each number from range $1 \leq i \leq j$ exactly one of rules that belong to set $\mathbf{R}_Y$, to obtain the sequence of all rules from set $\mathbf{R}_Y$: $R^1, \ldots, R^j$.

By definition of tableau rules 5.33, for each rule $R^i \in \mathbf{R}_Y$, there exists a finite number of n-tuples $\langle Y, X_1, \ldots, X_{n-1} \rangle$ in set R^i_Y. Therefore, each set R^i_Y, where $R^i \in \mathbf{R}_Y$, is finite — contains at most k of ordered n-tuples, for some $k \geq 1$. We take account of some sets $R^1_Y, \ldots, R^j_Y$, one for each $R^i \in \mathbf{R}_Y$.

We assign to each number from range $1 \leq i \leq k$ exactly one n-tuple from set R^i_Y, and denote given n-tuple as r_i to obtain the sequence of all n-tuples from set R^i_Y: $r_1, \ldots, r_k$.

Consequently, in any R^i_Y there exists a finite number of ordered n-tuples $\langle Y, X_1, \ldots, X_{n-1}\rangle$ that we can arrange in sequence: $r^i_1, \ldots, r^i_k$, for certain $k \geq 1$.

Next, we define a list of all n-tuples r^i_l from each R^i_Y, imposing a kind of lexicographical order on that list:

$$\underbrace{r^1_1, \ldots, r^1_m}_{R^1{}_Y}, \underbrace{r^2_1, \ldots, r^2_n}_{R^2{}_Y}, \ \ldots, \underbrace{r^j_1, \ldots, r^j_o}_{R^j{}_Y}, \text{ where } 1 \leq m, n, \ldots, o.$$

Such defined list of ordered n-tuples from set of expressions Y will be called Y*-list* and denoted as L_Y. Of course, there may exist multiple Y-lists. Still, there exists at least one Y-list that can be empty.

Let L_Y be Y-list and let $r_i \in L_Y$. We know that $r_i \in R^k{}_Y \subseteq R^k$, for some $k \leq j$. Let $r_i = \langle X_1, \ldots, X_n\rangle$. We shall state that ordered n-tuple $\langle Z_1, \ldots, Z_n\rangle$ is an *expansion* of r_i iff:

- $\langle Z_1, \ldots, Z_n\rangle \in R^k$
- for each $1 \leq l \leq n$ the following conditions are satisfied:
  1. $X_l \subset Z_l$.
  2. X_l is a set that is similar to $X_1 \cup (Z_l \setminus Z_1)$.

If $\langle Z_1, \ldots, Z_n\rangle$ is the considered expansion r_i, instead of $\langle Z_1, \ldots, Z_n\rangle$ we will write r'_i.

$(*)$ From definition of rules 5.29 (Closure under expansion), (Existence of core of rule), we know that for any $r_i = \langle X_1, \ldots, X_n\rangle$ that belongs to rule R and for any t-consistent set of expressions Z_1, such that $X_1 \subset Z_1$ and for each $1 < i \leq n$, X_i is not similar to any subset Z_1 that contains X_1, there exists such $r_j \in R$ that r_j is an expansion of r_i, where $r_j = \langle Z_1, \ldots, Z_n\rangle$, for some $Z_2, \ldots Z_n \subseteq \mathsf{Te}$.

Let L_Y be certain Y-list. By induction we define the *closure* of set Y under L_Y. $L_Y(Y)$ is a maximally long sequence of sets of expressions $Z_1, \ldots, Z_o$, such that for some $o \in \mathbb{N}$ and for any $1 \leq n \leq o$:

1. if $n = 1$, then $Z_n = Y$
2. if $n = 2$, then $Z_n = X_j$, where:
   a. r_1 is the first n-tuple in L_Y
   b. $r_1 = \langle Y, X_1, \ldots, X_n\rangle$, for $n \geq 1$
   c. $X_j = X_1$
3. if $n > 2$, then
   a. Z_{n-1} belongs to sequence $L_Y(Y)$
   b. Z_{n-1} is a consequence of expansion of certain m-tuple $r_l \in L_Y$ applied to Z_{n-2}, thus $r'_l = \langle Z_{n-2}, W_1, \ldots, W_m\rangle$, for $m \geq 1$, and $Z_{n-1} = W_1$

   and $Z_n = X_j$, where:

a. there exists r_{l+m}, for some $m \geq 1$, and it is the first element after r_l in L_Y such that:
b. r'_{l+m} is an expansion of r_{l+m}
c. $r'_{l+m} = \langle Z_{n-1}, X_1, \ldots, X_i \rangle$, for $i \geq 1$
d. $X_j = X_1$.

By definition of branch 5.35, each closure $L_Y(Y) := Z_1, \ldots, Z_n$, for some $n \in \mathbb{N}$, is a branch.

Now, let us investigate the initial set of expressions X. By virtue of the previous findings, we conclude:

- X is a finite set, so we have such branch $L^1_X(X) := X_1, \ldots, X_k$, for some X-list and some $k \in \mathbb{N}$ that:
- X_k is a finite set of expressions as set of tableau rules **R** is closed under finite sets 5.29 (Closure under finite sets).

Let us investigate a sequence of closures under some number of lists L^j — where $j \in \mathbb{N}$ — and assume that the last set from the last closure X_o is a finite set:

$$
\begin{array}{lr}
L^1_X(X) = X_1, \ldots, X_k & \text{for some } k \geq 1, \text{ where } k \in \mathbb{N} \\
L^2_{X_k}(X_k) = X_k, \ldots, X_l & \text{for some } l \geq k, \text{ where } k \in \mathbb{N} \\
. & \\
. & \\
. & \\
L^{j-1}_{X_{l+m}}(X_{l+m}) = X_{l+m}, \ldots, X_n & \text{for some } n \text{ and } m \in \mathbb{N}, \text{ where} \\
 & n \geq l+m \\
L^j_{X_n}(X_n) = X_n, \ldots, X_o & \text{for some } o \geq n, \text{ where } o \in \mathbb{N}.
\end{array}
$$

Since set of expressions X_o is finite, so we can define another branch $L^{j+1}_{X_o}(X_o) := X_o, \ldots, X_r$, for some X_o-list and certain $r \in \mathbb{N}$ such that:

1. by definition of addition of branches 5.37 and conclusion on addition of branches 5.38, $(((\ldots(L^1_X(X) \oplus L^2_{X_k}(X_k)) \oplus \ldots) \oplus L^{j-1}_{X_{l+m}}(X_{l+m})) \oplus L^j_{X_n}(X_n)) \oplus L^{j+1}_{X_o}(X_o)$ is a branch.
2. X_r is a finite set of expressions as set of tableau rules **R** is closed under finite sets by definition of tableau rules 5.29 (Closure under finite sets).

Consequently, for any $j \in \mathbb{N}$ there exists such branch $L^{j+1}_{X_m}(X_m)$ that $L^j_{X_l}(X_l) = X_l, \ldots, X_m$, $L^1_X(X) = X_1, \ldots, X_k$, and for some $k \leq \cdots \leq l \leq m \in \mathbb{N}$, $X_1 \subseteq X_k \subseteq \cdots \subseteq X_l \subseteq X_m$. We can extract all those branches:

$$\underbrace{X_1 = X, \ldots, X_k}_{L^1_X(X)}, \underbrace{X_k, \ldots, X_l}_{L^2_{X_k}(X_k)}, \underbrace{X_l, \ldots, X_m}_{L^3_{X_l}(X_l)}, \ldots$$

After removal of all duplicates of elements, we get a branch — let us call it χ — which begins with initial set X. We claim that branch χ is a maximal branch. There exist two options:

1. branch χ is finite in length
2. branch χ is infinite.

If the first option occurs, then from the construction of branch χ we know that there does not exist super-branch χ. And from conclusion 5.44 we deduce that branch χ is maximal.

Assume that the second case occurs — so branch χ is infinite in length. Let us investigate if χ is a maximal branch. Taking account of definition of maximal branch 5.42, assume that there exists such tableau rule $R \in \mathbf{R}$ and such sets of expressions $Y_1, \ldots Y_n \subseteq \mathsf{Te}$, for some $1 < n \in \mathbb{N}$, that:

- $\langle Y_1, \ldots, Y_n \rangle \in R$
- for some $1 \leq i$, $X_i = Y_1$ and $X_i \in \chi$.

We must demonstrate that there exists such index $j \in \mathbb{N}$ that for certain $1 < k \leq n$, some subset W of element of branch $X_j \in \chi$ is strongly similar to set Y_k.

From the construction of branch χ we know that $X_i \in L^k_{X_m}(X_m)$, for some $k \geq 1$ and $m \leq i$.

By definition of tableau rules 5.33 and construction of branch χ two cases are possible:

(a) $i = m$ and $R \in \mathbf{R}_{X_m}$

(b) $i > m$ and there exist:

1. some $l \in \mathbb{N}$, where $m < l$
2. subsequent sequence $L^{k+1}_{X_l}(X_l)$ and $R \in \mathbf{R}_{X_l}$.

Assume the first case, meaning $i = m$ and $R \in \mathbf{R}_{X_m}$. By virtue of construction of branch χ we have three options:

1. $X_{i+1} = Y_k$, for some $1 < k \leq n$
2. there exist: n-tuple $\langle W_1, \ldots, W_n \rangle \in R$ which is an expansion of the initial n-tuple $\langle Y_1, \ldots, Y_n \rangle$, and such element of branch χ $X_{i+o} = W_1$, for some $o \geq 1$ that $X_{i+o+1} = W_k$ and additionally certain subset W_k is strongly similar to set of expressions Y_k, for some $1 < k \leq n$

3. there exist: set of expressions $X_{i+o} \in \chi$, for some $o \geq 1$, and such rule $R' \in \mathbf{R}_{X_m}$ that certain ordered n-tuple $\langle X_{i+o-1}, Y_2, \ldots, Y_n \rangle \in R'$, for some $n \in \mathbb{N}$, $X_{i+o} = Y_{n_1}$, for some $1 < n_1 \leq n$, and certain subset X_{i+o} is strongly similar to Y_k, for some $1 < k \leq n$.

Case (b) consists of similar options, only that we consider such expansion of n-tuple $\langle Y_1, \ldots, Y_n \rangle \in R$ that the first set of that expansion contains set X_l.

Therefore, χ is a maximal branch. □

We still need a few additional concepts for the proof of the theorem on completeness. We will utilize them for demonstration of relationship between the existence of closed tableau an relation $\vDash$.

Now, we will define a successive concept relevant for the general theorem on completeness.

Definition 5.59 (Rules good for interpretations)**.** We shall state that set of rules $\mathbf{R}$ is *good* for set of interpretations $\mathbf{I}$ iff for any sets $X_1, \ldots, X_i \subseteq \mathsf{Te}$ (where $1 < i$), any interpretation $\mathfrak{I} \in \mathbf{I}$ and any rule $R \in \mathbf{R}$, if:

- $\langle X_1, \ldots, X_i \rangle \in R$
- $\mathfrak{I}$ is appropriate for X_1,

then $\mathfrak{I}$ is appropriate for X_j, for some $1 < j \leq i$.

We will use the above definition 5.59 for the proof of another lemma, assuming the property it defines. This lemma determines the relationship between the finite sets of formulas and the existence of maximal and open branches.

Lemma 5.60. *Let $X \subseteq \mathsf{For}$ be a finite set of formulas, $i \in \mathbb{N}$ and let $\mathfrak{I} \in \boldsymbol{I}$ be an interpretation of formulas. If set of rules $\mathbf{R}$ is good for set of interpretations $\boldsymbol{I}$ and $\mathfrak{I} \vDash X$, then there exists a maximal and open branch beginning with set $\{A^i : A \in X\}$.*

Proof. Take any finite set of formulas $X \subseteq \mathsf{For}$, any index $i \in \mathbb{N}$ and any interpretation of formulas $\mathfrak{I}$, and then assume that set of rules $\mathbf{R}$ is good for set of interpretations $\mathbf{I}$ and $\mathfrak{I} \vDash X$. We define the following set $\{A^i : A \in X\}$. Set $\{A^i : A \in X\}$ will be denoted as X^i.

Since $\mathfrak{I} \vDash X$, so from definition of relation $\Vdash_i$ 5.25 it follows that $\mathfrak{I} \Vdash X$. Moreover, by definition of tableau expressions 5.15, set X is t-consistent.

Consequently, due to definition 5.28, interpretation $\mathfrak{I}$ is appropriate for set X^i.

Now, indirectly assume that each maximal branch beginning with set X^i is closed.

As $\Phi(X^i)$ we will denote the set of all maximal branches beginning with set X^i. From fact 5.58, we know that for each finite set of tableau expressions Y there exists a maximal branch beginning with set Y. Thus, set $\Phi(X^i)$ is non-empty.

Since set $\Phi(X^i)$ is a set of all maximal branches beginning with set X^i, so it has the following property.

Now, assume that for some branch $\chi \in \Phi(X^i)$. Let for certain $n \in \mathbb{N}$ exist such rule $R \in \mathbf{R}$ and such m-tuple $\langle Z_1, \ldots, Z_m \rangle \in R$ that $\chi(n) = Z_1$ and $\chi(n+1) = Z_j$, for some $1 < j \leq m$.

Note that each set Z_j is a finite set of expressions since each rule expands the finite input set to the finite output set (by definition of tableau rules 5.29 (Closure under finite sets)), branch χ begins with finite set X^i, and we investigate its n-th element. Thus, from fact 5.58 we know that:

- for each set Z_j there exists maximal branch ϕ_j beginning with set X^i such that $\phi_i(n+1) = Z_j$.

$(*)$ Consequently, set $\Phi(X^i)$ contains such branches χ, $\phi_2, \ldots, \phi_m$ that $\chi = \phi_j$, for some $1 < j \leq m$.

Thus for any $n \in \mathbb{N}$, if there exist: such rule $R \in \mathbf{R}$, such m-tuple $\langle Z_1, \ldots, Z_m \rangle \in R$, and branch $\chi \in \Phi(X^i)$ such that $\chi(n) = Z_1$ and $\chi(n+1) = Z_j$, then there exists branch $\psi \in \Phi(X^i)$ such that $\psi(n+1) = Z_j$ and for any $k \leq n+1$, $\chi(k) = \psi(k)$.

$(**)$ By assumption, each branch that belongs to set $\Phi(X^i)$ is closed, thus by virtue of fact 5.40, each branch that belongs to set $\Phi(X^i)$ has a finite length of m, for some $m \in \mathbb{N}$.

From the initial assumption, we know that each of branches in set $\Phi(X^i)$ begins with set X^i.

Since interpretation $\mathfrak{I}$ is appropriate for set of expressions X^i, so due to the definition of interpretation appropriate for set of expressions 5.28, set X^i is not t-inconsistent. Hence, we get a conclusion that there are no branches of length one in set $\Phi(X^i)$.

Due to the assumption that set of rules $\mathbf{R}$ is good for set of interpretations $\mathbf{I}$ and definition 5.59, for any rule $R \in \mathbf{R}$ and any l-tuple $\langle Z_1, \ldots Z_l \rangle \in R$, if interpretation of formulas $\mathfrak{I}$ is appropriate for set Z_1, then it is also appropriate for some set Z_j, where $1 < j \leq l$, and by virtue of $(*)$, there exists branch $\chi \in \Phi(X^i)$ such that interpretation of formulas $\mathfrak{I}$ is appropriate for set $\chi(2)$ and $\chi(1) = X^i$.

The set of those branches that belong to $\Phi(X^i)$, and simultaneously interpretation of formulas $\mathfrak{I}$ is appropriate for their k-th element, will be denoted as $\Phi(X^i)_k$.

So, we have $\Phi(X^i) = \Phi(X^i)_1 \supseteq \Phi(X^i)_2 \neq \varnothing$.

Now, assume that for some $n \in \mathbb{N}$, where $n > 1$, set $\Phi(X^i)_{n-1} \supseteq \Phi(X^i)_n \neq \varnothing$. Since set $\Phi(X^i)_n$ is non-empty, so take some branch $\psi \in \Phi(X^i)_n$.

By assumption, interpretation of formulas $\mathfrak{I}$ is appropriate for set of expressions $\psi(n)$, so due to the definition of interpretation appropriate for set of expressions 5.28, set $\psi(n)$ is not t-inconsistent.

Due to the assumption that set of rules **R** is good for set of interpretations **I** and definition 5.59 which claims that for any rule $R \in \mathbf{R}$ and any l-tuple $\langle Z_1, \dots Z_l \rangle \in R$, if interpretation of formulas $\mathfrak{I}$ is appropriate for set Z_1, then it is also appropriate for some set Z_j, where $1 < j \leq l$, and $(*)$, there exists branch $\phi \in \Phi(X^i)_{n+1}$ such that interpretation $\mathfrak{I}$ is appropriate for set $\phi(n+1)$ and $\phi \in \Phi(X^i)_n$. Thus, $\Phi(X^i)_n \supseteq \Phi(X^i)_{n+1}$ and $\Phi(X^i)_{n+1} \neq \varnothing$.

Therefore, for each $k \in \mathbb{N}$:

$$\Phi(X^i) = \Phi(X^i)_1 \supseteq \Phi(X^i)_2 \supseteq \cdots \supseteq \Phi(X^i)_k \supseteq \dots$$

We take the intersection of all those sets $\Phi(X^i)_k$, where $k \in \mathbb{N}$. Intersection $\bigcap\{\Phi(X^i)_k : k \in \mathbb{N}\} = \Phi$ is non-empty as for each k, subset $\Phi(X^i)_k$ is also non-empty. So, set Φ includes at least one branch χ. That branch is maximal and begins with set X^i since $\Phi \subseteq \Phi(X^i)$.

But, branch χ is infinite which contradicts conclusion $(**)$. □

We will now move on to the final, auxiliary relationship between the tableau concepts.

Lemma 5.61. *Let $X \subseteq \mathsf{For}$ be a finite set of formulas, $A \in \mathsf{For}$ and $i \in \mathbb{N}$. If there exists a maximal and open branch beginning with set $\{B^i : B \in X \cup \{f(A)\}\}$, then each complete tableau $\langle X, A, \Phi \rangle$ is open.*

Proof. Take finite set $X \subseteq \mathsf{For}$, any formula $A \in \mathsf{For}$ and index $i \in \mathbb{N}$ such that there exists a maximal and open branch beginning with set $\{B^i : B \in X \cup \{f(A)\}\}$. We will denote that branch by letter ϕ, and set $\{B^i : B \in X \cup \{f(A)\}\}$, for simplicity, will be denoted by X^i.

$(*)$ Since branch ϕ is open, so no element ϕ is a t-inconsistent set, by definition 5.39.

$(**)$ Since branch ϕ is maximal and open, so for any rule $R \in \mathbf{R}$, any $n \in \mathbb{N}$ and any element $Y \in \phi$, if $\langle Y, Y_1, \dots, Y_n \rangle \in R$, then there exists some element $Z \in \phi$ such that some subset $W \subseteq Z$ is a set strongly similar to set Y_i, for certain $1 \leq i \leq n$, by definition of maximal branch 5.42.

Now, we indirectly assume that there exists complete and closed tableau $\langle X, A, \Psi \rangle$.

Since tableau $\langle X, A, \Psi \rangle$ is complete, so Ψ is such subset of set of all maximal branches that $\langle X, A, \Psi \rangle$ is a complete tableau, by definition of complete tableau 5.50.

Since tableau $\langle X, A, \Psi \rangle$ is closed, so each branch that belongs to Ψ, is closed, by definition of closed tableau 5.51. For certain $k \in \mathbb{N}$, each of these branches:

- begins with set $X^k = \{B^k : B \in X \cup \{f(A)\}\}$, by definition of tableau 5.48
- and its last element is a t-inconsistent set of expressions, by definition of closed tableau 5.51.

We intend to show that there exists some open branch ψ in set of branches Ψ, which contradicts the assumption that $\langle X, A, \Psi \rangle$ is a closed tableau. To this end, we will apply the induction through the branch length in order to construct infinite branches beginning with set X^k. The construction method for such branches will be denoted as $(\dagger)$.

Consider the first element of each branch contained in set of branches Ψ. It is set $X_1 = X^k = \{B^k : B \in X \cup \{f(A)\}\}$. X_1 is a set of expressions similar — within the meaning of definition of similarity 5.23 — to set $X^i = \{B^i : B \in X \cup \{f(A)\}\}$. Since $X^i \in \phi$ and branch ϕ is open, so X^i and X_1 are t-consistent, by definition 5.23.

Nevertheless, due to the fact that Ψ is a set of closed branches and the considered tableau $\langle X, A, \Psi \rangle$ is complete, there must exist a tableau rule $R \in \mathbf{R}$ such that $\langle X_1, Z_2, \ldots, Z_l \rangle \in R$, where $l < 1$, and for each $1 < j \leq l$ there exists such branch in set Ψ that Z_j belongs to that branch, by definition of complete tableau 5.50.

Nonetheless, certain set Z_m — for $1 < m \leq l$ — must be t-consistent. Because due to definition of tableau rules 5.29, there exists such l-tuple that $\langle Y_1, \ldots, Y_l \rangle \in R$, where Z_m is a similar set — within the meaning of definition of similarity 5.23 — to some set $W_m \subseteq Y_m$ and it is t-consistent, since $Y_m \subseteq U \in \phi$, for certain $U \subseteq \mathsf{Te}$, by virtue of the fact that ϕ is an open, by $(*)$, and maximal branch, by $(**)$. Set Z_m will be denoted as X_2, while element W_m as X_2^*.

Therefore, for number 1 there exist such branches ψ_1, $\psi_2 \in \Psi$ that:

- $X_1 \in \psi_1$
- set X_2 originated by the application of certain rule $R \in \mathbf{R}$ to set X_1, ultimately producing a second element of branch $\psi_2 \in \Psi$
- $X_2 \in \psi_2$
- X_2 is a t-consistent set
- $X_1 \subset X_2$
- for some $j \in \mathbb{N}$, set $X_2^* \subseteq X_j \in \phi$, where set X_2^* is similar, in the sense of definition of similarity 4.16, to set X_2.

Now, assume that for certain $n \in \mathbb{N}$ there exist such branches $\psi_1, \ldots, \psi_n \in \Psi$ that:

- for any $1 < j \leq n$, set X_j originated by the application of certain rule $R \in \mathbf{R}$ to set X_{j-1}, ultimately producing j-th element of branch $\psi_j \in \Psi$

- $X_n \in \psi_n$
- X_n is a t-consistent set
- $X_1 \subset X_2 \subset \ldots \subset X_n$
- for some $i \in \mathbb{N}$, set $X_n^* \subseteq X_i \in \phi$, where X_n^* is similar, in the sense of definition of similarity 4.16, to set X_n.

Nevertheless, due to the fact that Ψ is a set of closed branches, the considered tableau $\langle X, A, \Psi \rangle$ is complete and set X_n is a t-consistent set, there must exist a tableau rule $R \in \mathbf{R}$ such that $\langle X_n, Z_2, \ldots, Z_l \rangle \in R$, where $l > 1$, and for each $1 < j \leq l$ there exists such branch in set Ψ that Z_j belongs to that branch, by definition of complete tableau 5.50.

Nonetheless, certain set Z_m — for $1 < m \leq l$ — must be t-consistent. Because due to definition of tableau rules 5.29, there exists such l-tuple that $\langle Y_1, \ldots, Y_l \rangle \in R$, where Z_m is a similar set — within the meaning of definition of similarity 5.23 — to some set $W_m \subseteq Y_m$ and it is t-consistent since $Y_m \subseteq U \in \phi$, for certain $U \subseteq \mathsf{Te}$, by virtue of the fact that ϕ is an open $(*)$ and maximal branch $(**)$. Set Z_m will be denoted as X_{n+1}, while element W_m as X_{n+1}^*.

Thus, for any $n \in \mathbb{N}$, there exist such branches $\psi_1, \ldots, \psi_n, \psi_{n+1} \in \Psi$ that:

1. for any $1 < j \leq n+1$, set X_j originated by the application of certain rule $R \in \mathbf{R}$ to set X_{j-1}, ultimately producing j-th element of branch $\psi_j \in \Psi$
2. $X_{n+1} \in \psi_{n+1}$
3. X_{n+1} is a t-consistent set
4. $X_1 \subset X_2 \subset \ldots \subset X_n \subset X_{n+1}$
5. for some $i \in \mathbb{N}$, set $X_{n+1}^* \subseteq X_i \in \phi$, where X_{n+1}^* is similar, within the meaning of definition of similarity 5.23 — to set X_{n+1}.

Set of all sets that originate this way $X_1 \subset X_2 \subset \ldots \subset X_n \subset X_{n+1} \subset \ldots$ will be denoted as $\mathbf{X}$. Set $\mathbf{X}$ contains at least one branch ψ such that for any $i \in \mathbb{N}$, if $X_i \in \psi$, then there exists set $X_i \in \mathbf{X}$.

Branch ψ can be defined through the specification of such minimal subset of $\mathbf{X}$, set $\mathbf{X}'$ that:

- $X_1 \in \mathbf{X}'$
- for any $i \in \mathbb{N}$, if $X_i \in \mathbf{X}'$, then exactly one $X_{i+1} \in \mathbf{X}'$.

Branch ψ is infinite, and as a consequence of conclusion 5.40 it is an open branch.

Since set X_1, the first element of branch ψ, is equal to set X^k, and moreover for any element $X_i \in \psi$, where $i > 1$, there exist rule $R \in \mathbf{R_{S5}}$ and n-tuple $\langle Y_1, \ldots Y_n \rangle \in R$ such that:

- $Y_1 = X_{i-1}$
- $X_i = Y_k$, for certain $1 < k \leq n$
- for each $1 < j \leq n$, if $j \neq k$, then there exists branch $\psi' \in \Psi$ such that for some Z_l, where $1 \leq l$, $Z_l \in \psi'$, $Z_l = Y_1$ and $Z_{l+1} = Y_j$,

so $\langle X, A, \Psi \cup \{\psi\} \rangle$ by definition of tableau 5.48 is a tableau for pair $\langle X, A \rangle$.

However, branch ψ does not belong to set Ψ because tableau $\langle X, A, \Psi \rangle$, contrary to the assumption, would not be a closed tableau.

Let us now consider the question whether or not set $\Psi \cup \{\psi\}$ is a redundant superset of set Ψ, in the light of definition of redundant variant of branch 5.49. Let us now carry out the following argument.

$(\dagger\dagger)$ Assume that branch ψ is a redundant variant of some branch $\psi' \in \Psi$ different from ψ. For a certain minimal $1 \leq i \in \mathbb{N}$:

- there exists such rule $R \in \mathbf{R}$ and such n-tuple $\langle X_1, \ldots, X_n \rangle \in R$ that $\psi'(i) = X_1$ and $\psi'(i+1) = X_j$, for certain $1 < j \leq n$
- there exists rule $R \in \mathbf{R}$ and such m-tuple $\langle Y_1, \ldots, Y_m \rangle \in R'$, where $n < m$, that $X_1 = \psi(i) = Y_1$ and:
  1. $\psi(i+1) = Y_k$, for certain $1 < k \leq m$
  2. for any $1 < l \leq n$ there exists such $1 < o \leq m$ that $o \neq k$ and $X_l = Y_o$ and there exists such branch $\psi'' \in \Psi$ that $\psi''(i+1) = Y_o$.

But since branch ϕ is open and maximal (assumptions $(*)$ and $(**)$), so also some element $X_l = \psi''(i+1)$ for some branch $\psi'' \in \Phi$ is t-consistent because it is similar to some set of expressions W included in some element of branch ϕ.

Therefore, we can construct yet another infinite and open branch $X_1, \ldots, X_{j+1}$, making use of construction $(\dagger)$ which again, for at least successive element, i.e. $X_{j+1} = X_l$, by virtue of reasoning analogous to $(\dagger\dagger)$ is t-consistent and it is not a redundant variant of any sub-branch of any branch from set Ψ.

So, by application of induction and steps $(\dagger)$ and $(\dagger\dagger)$ we get an infinite branch — call it χ — and, consequently, open which is not a redundant variant of any branch that belongs to set of branches Ψ and begins with set X_1.

Since Ψ, by assumption, contains closed branches, so $\chi \notin \Psi$. Since set X_1, the first element of branch χ, is equal to set X^k, and moreover for any element $X_i \in \chi$, where $i > 1$, there exists such rule $R \in \mathbf{R}$ and such n-tuple $\langle Y_1, \ldots Y_n \rangle \in R$ that:

- $Y_1 = X_{i-1}$
- $X_i = Y_k$, for certain $1 < k \leq n$
- for each $1 < j \leq n$, if $j \neq k$, then there exists branch $\psi \in \Psi$ such that for some Z_l, where $1 \leq l$, $Z_l \in \psi$, $Z_l = Y_1$ and $Z_{l+1} = Y_j$,

so $\langle X, A, \Psi \cup \{\chi\} \rangle$ by virtue of definition of tableau 5.48 is a tableau for pair $\langle X, A \rangle$.

Thus, $\langle X, A, \Psi\rangle$ is not a complete tableau which contradicts the initial assumption. □

Summing up the definitions, lemmas and facts that we have presented so far, we move on to the general theorem on completeness for the tableau systems constructed using the method presented in the book.

Theorem 5.62 (General theorem on completeness). *If:*

1. *set of interpretations* **I** *is good for set of tableau rules* **R**
2. *set of tableau rules* **R** *is good for set of interpretations* **I**

then for any $X \subseteq \mathsf{For}$, $A \in \mathsf{For}$ *the below statements are equivalent:*

- $X \vDash A$
- $X \rhd A$
- *there exists finite subset* $Y \subseteq X$ *and closed tableau* $\langle Y, A, \Phi\rangle$.

Proof. We assume 1., 2. and take any $X \subseteq \mathsf{For}$, $A \in \mathsf{For}$. We must prove three implications.

(a) $X \vDash A \Longrightarrow X \rhd A$.

Assume that $X \not\rhd A$. Hence, for any finite $Y \subseteq X$ there exists an open and maximal branch beginning with set $Y^i \cup f(A)^i$ — for some $i \in \mathbb{N}$ — by definition of branch consequence 5.46. By lemma 5.57 (On the existence of open and maximal branch), there exists a closure of set $X^i \cup f(A)^i$ under rules **R** which constitutes a maximal and open branch ψ.

By assumption 1, we know that there exists interpretation $\mathfrak{I} \in \mathbf{I}$ generated by ψ and $\mathfrak{I} \vDash X \cup \{f(A)\}$. Therefore $\mathfrak{I} \vDash X$ and $\mathfrak{I} \nvDash A$, by definition 5.8. Consequently $X \nvDash A$.

(b) $X \rhd A \Longrightarrow$ there exists finite subset $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi\rangle$.

$(*)$ Assume that for each finite subset $Y \subseteq X$ all tableaux $\langle Y, A, \Phi\rangle$ are open.

Take any finite subset $Y \subseteq X$. Note that from fact 5.58 $(**)$ it follows that for any finite set of tableau expressions there exists a maximal branch which begins with that set. Therefore for any index $i \in \mathbb{N}$ there exists maximal branch beginning with set $Y^i \cup \{f(A)^i\}$.

Take any index $i \in \mathbb{N}$. So, set of maximal branches Φ^i beginning with set $Y^i \cup \{f(A)^i\}$ is non-empty. What is more, by $(**)$ set Φ^i contains at least one such subset Φ that ordered triple $\langle Y, A, \Phi\rangle$ is a complete tableau.

Due to assumption $(*)$ tableau $\langle Y, A, \Phi \rangle$ is open.

Since $\langle Y, A, \Phi \rangle$ is open and complete, so Φ contains maximal and open branch ϕ which begins with set $Y^i \cup \{f(A)^i\}$.

Since Y is any finite subset X and i is any index, so for any finite subset $Y \subseteq X$ and any index $i \in \mathbb{N}$, there exists some maximal and open branch ψ beginning with set $Y^i \cup \{f(A)^i\}$.

Consequently, there does not exist such finite subset $Y \subseteq X$ and such index $i \in \mathbb{N}$ that each maximal branch beginning with $Y^i \cup \{f(A)^i\}$ is closed. Therefore — by definition 5.46 — $X \not\triangleright A$.

(c) there exists finite set $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi \rangle \Longrightarrow X \vDash A$.

Assume that $X \nvDash A$. So, by definition of relation of semantic consequence 5.10, there exists such interpretation of formulas $\mathfrak{I}$ that $\mathfrak{I} \vDash X$ and $\mathfrak{I} \nvDash A$. Thus $\mathfrak{I} \vDash f(A)$, and consequently $\mathfrak{I} \vDash X \cup \{f(A)\}$.

Hence, for any finite subset $Y \subseteq X$, also $\mathfrak{I} \vDash Y \cup \{f(A)\}$.

Take any finite subset $Y' \subseteq X$. From lemma 5.60 and assumption 2 (set of tableau rules **R** is good for set of interpretations **I**), we get a conclusion that for any $i \in \mathbb{N}$ there exists maximal and open branch beginning with set $\{B^i : B \in Y' \cup \{f(A)\}\}$.

And from lemma 5.61 we know that each complete tableau $\langle Y', A, \Phi \rangle$ is open. Since Y' was an arbitrary finite subset of set of formulas X, so there is no finite set $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi \rangle$. □