6 Examples of applications

6.1 Introductory remarks

In this chapter, we will show exemplary applications of the general tableau concepts we defined in Chapter Five. Thanks to the general concepts we have at our command, and their interrelationships we have demonstrated, we can significantly shorten the construction of a complete tableau system.

We already have the general concepts:

- set of tableau rules
- branches
- closed/open branch
- relation of branch consequence
- tableau
- open/closed tableau.

We also know that there exists a general connection between a properly defined set of tableau rules **R** and properly defined semantics. Further work on the construction of the complete tableau system must therefore focus solely on defining the detailed concepts of tableau system in such a way that the general conditions are met — if that is the case, then we get a tableau system that is complete in terms of the initial semantics.

In the next four subchapters, we will describe three different applications. The first one will be of a detailed nature. We will consider an example of the logic of categorical propositions with modalities *de re*. We will define the basic concepts and then show that they meet the sufficient conditions for the general theorem on completeness 5.62, which will allow us to reach the conclusion that the defined tableau system is adequate to the initial semantics. This kind of application of the general theorem on completeness can be considered paradigmatic, because the theorem is intended primarily to shorten the construction of complete tableau systems when we want to construct a tableau system for a logic that is already semantically defined.

The second application, described in subsequent subchapter, has a general character. In this subchapter, we describe how to apply the general tableau concepts and general tableau theorem to the entire class of logics defined with the same type of semantics. Using an example of modal logics specified with the semantics of possible worlds, we will show how to obtain a less general theorem on completeness, specified for this class of logics. It allows even simpler proof of the completeness of specific tableau systems, because some more general properties are fulfilled by the entire class of modal logics specified with the semantics of possible worlds. Similar general applications can occur in all cases where we consider the classes of logic defined with a common type of semantics.

The next subchapter is devoted to the concept of a tableau system. We will try to define the general concept of tableau system and show what benefits to the investigation of dependencies between tableau systems brings the way the book describes tableau systems.

In the last subchapter, we will outline the transition between the formalised tableaux and standard tableaux/trees. Therefore, we will try to show that the approach presented in the book corresponds to the standard approach, as to the practical construction of proof itself, while at the same time emphasizing the general nature of concepts that the standard approach does not bear.

6.2 Tableau system for Modal Term Logic *de re*

We will now turn to the logic of categorical propositions *de re*.[1] It is an extension of logic **TL** with new categorical propositions with modalities in the interpretation *de re*. This logic will be called Modal Term Logic *de re*, for short **MTL**.

When defining set of formulas of **MTL** on the right hand side we will provide schemes of propositions in English which correspond to particular formulas and may occur in the reasonings described by **MTL**.

6.2.1 Language

Let us begin with the alphabet of **MTL**.

Definition 6.1 (Alphabet of **MTL**). Alphabet of Modal Term Logic is made up by the sum of the following sets:

- set of logical constants $\mathsf{Lc} = \{\mathbf{a}, \mathbf{i}, \mathbf{e}, \mathbf{o}, \mathbf{a}^{\Diamond}, \mathbf{i}^{\Diamond}, \mathbf{e}^{\Diamond}, \mathbf{o}^{\Diamond}, \mathbf{a}^{\Box}, \mathbf{i}^{\Box}, \mathbf{e}^{\Box}, \mathbf{o}^{\Box}\}$
- set of name letters $\mathsf{Ln} = \{P^1, Q^1, R^1, P^2, Q^2, R^2, \dots\}$.

Even though the set of name letters is infinite and contains indexed letters, practicably we will use a finite number of the following letters: *P*, *Q*, *R*, *S*, *T*, *U*, treating them as metavariables ranging over set Ln.

Let us now proceed to the definition of formula of **MTL**. Modal Term Logic is defined on the following set of formulas.

1 The idea for the system presented here was offered first in [14]. A simplified and better developed in some respects version of this material is presented in [22]. Also other variants of syllogistic are studied there.

Definition 6.2 (Formula of **MTL**)**.** Set of formulas of **MTL** is the smallest set containing the following expressions.

- $P\mathbf{a}Q$ — *Each P is Q.*
- $P\mathbf{i}Q$ — *Some P is Q.*
- $P\mathbf{e}Q$ — *No P is Q.*
- $P\mathbf{o}Q$ — *Some P is not Q.*
- $P\mathbf{a}^{\Box}Q$ — *Each P must be Q.* *Each P is necessarily Q.*
- $P\mathbf{i}^{\Box}Q$ — *Some P must be Q.* *Some P is necessarily Q.*
- $P\mathbf{e}^{\Box}Q$ — *No P may be Q.* *No P is possibly Q.*
- $P\mathbf{o}^{\Box}Q$ — *Some P must not be Q.* *Some P is not possibly Q.*
- $P\mathbf{a}^{\Diamond}Q$ — *Each P may be Q.* *Each P is possibly Q.*
- $P\mathbf{i}^{\Diamond}Q$ — *Some P may be Q.* *Some P is possibly Q.*
- $P\mathbf{e}^{\Diamond}Q$ — *No P must be Q.* *No P is necessarily Q.*
- $P\mathbf{o}^{\Diamond}Q$ — *Some P may not be Q.* *Some P is not necessarily Q.*

where $P, Q \in \mathsf{Ln}$.
We specify set of formulas as $\mathsf{For}_{\mathbf{MTL}}$, and its elements will be called *formulas*.

6.2.2 Semantics

Let us define the concept of model. We will use semantics without possible worlds.

Definition 6.3 (Model for language of **MTL**)**.** *Model* will be called ordered quadruple $\mathfrak{M}_{\mathbf{MTL}} = \langle D, d^{\Box}, d, d^{\Diamond} \rangle$, where:

1. D is a set
2. $d^{\Box}$, d, $d^{\Diamond}$ are such functions from set of name letters Ln in set $P(D)$ that for any name letter $P \in \mathsf{Ln}$, $d^{\Box}(P) \subseteq d(P) \subseteq d^{\Diamond}(P)$.

Remark 6.4. The proposed concept of model expresses the following intuitions. Functions $d^{\Box}, d, d^{\Diamond}$ assign to each name letter P those sets of objects that are — respectively — necessarily P-s, are P-s, and can be P-s. The objects that are necessarily P-s, are also simply P-s and are contained in the set of those objects that can be P-s. Hence, we have a chain of inclusions: $d^{\Box}(P) \subseteq d(P) \subseteq d^{\Diamond}(P)$.

Now, we proceed to the concept of truth in model.

Definition 6.5 (Truth in model). Let $\mathfrak{M}_{\mathbf{MTL}} = \langle D, d^{\Box}, d, d^{\Diamond} \rangle$ be a model and let $A \in \mathsf{For}_{\mathbf{MTL}}$. We shall state that formula A is *true in model* $\mathfrak{M}_{\mathbf{MTL}}$ (for short $\mathfrak{M}_{\mathbf{MTL}} \vDash A$) iff for some name letters P, $Q \in \mathsf{Ln}$, one of the below conditions is met:

1. $A = P\mathbf{a}Q$ and $d(P) \subseteq d(Q)$
2. $A = P\mathbf{i}Q$ and $d(P) \cap d(Q) \neq \varnothing$
3. $A = P\mathbf{e}Q$ and $d(P) \cap d(Q) = \varnothing$
4. $A = P\mathbf{o}Q$ and $d(P) \nsubseteq d(Q)$
5. $A = P\mathbf{a}^{\Box}Q$ and $d(P) \subseteq d^{\Box}(Q)$
6. $A = P\mathbf{i}^{\Box}Q$ and $d(P) \cap d^{\Box}(Q) \neq \varnothing$
7. $A = P\mathbf{e}^{\Box}Q$ and $d(P) \cap d^{\Diamond}(Q) = \varnothing$
8. $A = P\mathbf{o}^{\Box}Q$ and $d(P) \nsubseteq d^{\Diamond}(Q)$
9. $A = P\mathbf{a}^{\Diamond}Q$ and $d(P) \subseteq d^{\Diamond}(Q)$
10. $A = P\mathbf{i}^{\Diamond}Q$ and $d(P) \cap d^{\Diamond}(Q) \neq \varnothing$
11. $A = P\mathbf{e}^{\Diamond}Q$ and $d(P) \cap d^{\Box}(Q) = \varnothing$
12. $A = P\mathbf{o}^{\Diamond}Q$ and $d(P) \nsubseteq d^{\Box}(Q)$.

If for any propositional letters P, $Q \in \mathsf{Ln}$ none of the conditions is met, then we shall state that formula A is *false in model* $\mathfrak{M}_{\mathbf{MTL}}$ (for short $\mathfrak{M}_{\mathbf{MTL}} \nvDash A$).
Let $X \subseteq \mathsf{For}_{\mathbf{MTL}}$. We shall state that set of formulas X is *true in model* $\mathfrak{M}_{\mathbf{MTL}}$ (for short: $\mathfrak{M}_{\mathbf{MTL}} \vDash X$) iff for any formula $A \in X$, $\mathfrak{M}_{\mathbf{MTL}} \vDash A$. We shall state that set of formulas X is *false in model* $\mathfrak{M}_{\mathbf{MTL}}$ (for short: $\mathfrak{M}_{\mathbf{MTL}} \nvDash X$) iff it is not the case that $\mathfrak{M}_{\mathbf{MTL}} \vDash X$.

Remark 6.6. The semantics for modal syllogistic we offer in this study refers to the semantics presented in studies of F. Johnson [16] and S. K. Thomason [26] i [27].[2]

In [16] Johnson introduced model in form $\langle W, V^e, V^a, V^a_c, V^e_c \rangle$ where V^e, V^a, V^a_c, and V^e_c are functions that assign subsets of set W to name letters, where W is treated as a set of all objects ("world"); $V^e(S)$ is a set of objects that essentially are S-s; $V^a(S)$ is a set of objects that accidentally are S-s; $V^e_c(S)$ is a set of objects that essentially are non-S-s; $V^a_c(S)$ is a set of objects that accidentally are non-S-s. Furthermore, an auxiliary function V was adopted as $V(S) = V^e(S) \cup V^a(S)$, that is $V(S)$ is a set of all S-s. Therefore, our D, d, and $d^{\Box}$ are Johnson's W, V, and V^e

2 The application of this type of semantics was launched in [14].

respectively. Moreover, our function $d^{\Diamond}$ can be defined with formula $d^{\Diamond}(S) := W \setminus V^e_c(S)$; set of all those objects that are not essentially non-S-s, i.e. can be S-s.[3]

Our interpretation of proposition $S\mathbf{a}^{\Box}P$ corresponds to the interpretation of Johnson: $V(S) \subseteq V^e(P)$; and for $S\mathbf{e}^{\Box}P$ we have: $d(S) \cap d^{\Diamond}(P) = \varnothing$, i.e. $V(S) \cap (W \setminus V^e_c(P)) = \varnothing$ iff $V(S) \subseteq V^e_c(P)$. For the other propositions [16] adopted different interpretation that ours, in order to reproduce the modal syllogistic of Aristotle.[4]

In [26] Thomason used a semantics based on the ordered quadruples in form $\langle W, \mathrm{Ext}, \mathrm{Ext}^+, \mathrm{Ext}^- \rangle$, where W is a set of objects and Ext, Ext^+, and Ext^- are functions that assign subsets of set W to name letters and meet the following conditions: $\varnothing \neq \mathrm{Ext}^+(x) \subseteq \mathrm{Ext}(x)$ and $\mathrm{Ext}^-(x) \cap \mathrm{Ext}(x) \neq \varnothing$, for each letter x. Thomason's W, Ext and Ext^+ correspond to our D, d and $d^{\Box}$. Furthermore, his $\mathrm{Ext}^-(S)$ is a set of objects that cannot be S-s. We can express that set through our $D \setminus d^{\Diamond}(S)$. We have $\mathrm{Ext}(S) \subseteq (W \setminus \mathrm{Ext}^-(S))$, that is $d(S) \subseteq d^{\Diamond}(S)$. Obviously, Ext, Ext^+ and Ext^- correspond to functions V, V^e, and V^e_c from [16].

Denotation 6.7. Let $\bullet \in \{\Box, \Diamond\}$. Let us adopt denotation: if $\bullet = \Box$, then $\bullet' = \Diamond$, and if $\bullet = \Diamond$, then $\bullet' = \Box$.

Now, we will define a function that assigns a contradictory formula to each formula.

Definition 6.8 (Contradictory formulas)**.** Let $\circ\colon \mathsf{For}_{\mathbf{MTL}} \longrightarrow \mathsf{For}_{\mathbf{MTL}}$ be a function specified for any P, $Q \in L\mathsf{n}$ and $\bullet \in \{\Box, \Diamond\}$ with the following conditions:

1. $\circ(P\mathbf{a}Q) = P\mathbf{o}Q$
2. $\circ(P\mathbf{i}Q) = P\mathbf{e}Q$
3. $\circ(P\mathbf{e}Q) = P\mathbf{i}Q$
4. $\circ(P\mathbf{o}Q) = P\mathbf{a}Q$
5. $\circ(P\mathbf{a}^{\bullet}Q) = P\mathbf{o}^{\bullet'}Q$
6. $\circ(P\mathbf{i}^{\bullet}Q) = P\mathbf{e}^{\bullet'}Q$
7. $\circ(P\mathbf{e}^{\bullet}Q) = P\mathbf{i}^{\bullet'}Q$
8. $\circ(P\mathbf{o}^{\bullet}Q) = P\mathbf{a}^{\bullet'}Q$.

Directly from definition of truth in model 6.5 and definition of function $\circ$ 6.8 we get a conclusion.

Corollary 6.9. *For any model $\mathfrak{M}_{\mathbf{MTL}}$ and any formula A, $\mathfrak{M}_{\mathbf{MTL}} \models A$ iff $\mathfrak{M}_{\mathbf{MTL}} \not\models \circ(A)$*

3 Function V^a_c is superfluous in this semantics. This is also evident in studies of Thomason [26] and [27] who disregards that function.

4 For instance, propositions $S\mathbf{i}^{\Box}P$ and $P\mathbf{i}^{\Box}S$ are to be equivalent in this semantics.

Note that each model $\mathfrak{M}_{\mathbf{MTL}}$ can be identified with interpretation $\mathfrak{I}$ — in accordance with definition of general interpretation of formulas 5.8. Take any model $\mathfrak{M}_{\mathbf{MTL}}$ and define set of formulas $X_{\mathfrak{M}_{\mathbf{MTL}}} = \{A \in \mathsf{For}_{\mathbf{MTL}} : \mathfrak{M}_{\mathbf{MTL}} \vDash A\}$. Function $\circ$ is injective and for any formula B, $B \in X_{\mathfrak{M}_{\mathbf{MTL}}}$ iff $\circ(B) \notin X_{\mathfrak{M}_{\mathbf{MTL}}}$, by conclusion 6.9.

We define conventionally the relation of semantic consequence.

Definition 6.10 (Semantic consequence relation)**.** Let set $X \subseteq \mathsf{For}_{\mathbf{MTL}}$ and $A \in \mathsf{For}_{\mathbf{MTL}}$. We shall state that formula A *follows from* set of formulas X (for short: $X \vDash A$) iff for any model $\mathfrak{M}_{\mathbf{MTL}}$, if $\mathfrak{M}_{\mathbf{MTL}} \vDash X$, then $\mathfrak{M}_{\mathbf{MTL}} \vDash A$. We shall state that from set of formulas X *does not follow* formula A (for short: $X \nvDash A$) iff it is not the case that $X \vDash A$.

Pair $\langle \mathsf{For}_{\mathbf{MTL}}, \vDash \rangle$ is a semantically defined logic, in accordance with general definition 5.11. This implies that relation of semantic consequence $\vDash$ both is unambiguously determined by set of all models $\mathfrak{M}_{\mathbf{MTL}}$ for $\mathsf{For}_{\mathbf{MTL}}$ of set of formulas **MTL**, and unambiguously determines set of all models $\mathfrak{M}_{\mathbf{MTL}}$ for $\mathsf{For}_{\mathbf{MTL}}$ of set of formulas **MTL**, by fact 5.12.

6.2.3 Tableau expression

Before we move on to the definition of Te — set of tableau expression for **MTL** in accordance with general definition of tableau expression 5.15 — let us define several auxiliary concepts.

Definition 6.11 (Expressions)**.** Set of expressions Ex is the union of the following sets.

- $\{A^i : A \in \mathsf{For}_{\mathbf{MTL}}, i \in \mathbb{N}\}$
- $\{P_{+i} : P \in \mathsf{Ln}, i \in \mathbb{N}\}$
- $\{P_{-i} : P \in \mathsf{Ln}, i \in \mathbb{N}\}$
- $\{P^{\bullet}_{+i} : P \in \mathsf{Ln}, i \in \mathbb{N}\}$, where $\bullet \in \{\Box, \Diamond\}$
- $\{P^{\bullet}_{-i} : P \in \mathsf{Ln}, i \in \mathbb{N}\}$, where $\bullet \in \{\Box, \Diamond\}$.

By virtue of definition 6.11, the following conclusion occurs

Corollary 6.12. *There exists function $g : \mathsf{For}_{\mathbf{MTL}} \longrightarrow P(\mathsf{Ex})$, defined with condition: for any formula $A \in \mathsf{For}_{\mathbf{MTL}}$, $g(A) = \{A^1, A^2, A^3, \dots\}$, where $\{A^1, A^2, A^3, \dots\} \subseteq \mathsf{Ex}$.*

Remark 6.13. Practically, in the case of **MTL**, each $A^i \in g(A)$ will be simply identified with formula A, since in the tableau proof each formula will be self-represented.

Next, based on set Ex we define the concept of inconsistent set of expressions that ultimately will correspond to the concept of tableau inconsistent set of expressions.

Definition 6.14 (Inconsistent set of expressions)**.** Let $X \subseteq \mathsf{Ex}$. We shall state that set X is an *inconsistent set of expressions* iff X meets one of the following conditions:

1. $A \in X$ and $\circ(A) \in X$, for some $A \in \mathsf{For}_{\mathbf{MTL}}$
2. $P_{+i} \in X$ and $P_{-i} \in X$, for some $P \in \mathsf{Ln}$ and some $i \in \mathbb{N}$
3. $P^{\bullet}_{+i} \in X$ and $P^{\bullet}_{-i} \in X$, for some $P \in \mathsf{Ln}$, some $i \in \mathbb{N}$ and $\bullet \in \{\Diamond, \Box\}$.

We shall state that set X is a *consistent set of expressions* iff X is not an inconsistent set of expressions.

Based on definition of model 6.3, conclusion 6.9 and definition 6.14, we get another conclusion.

Corollary 6.15. *Let* $X \subseteq \mathsf{For}_{MTL}$ *and* $i \in \mathbb{N}$. *If there exists model* $\mathfrak{M}_{MTL}$ *such that* $\mathfrak{M}_{MTL} \models X$, *then set* $\{x^i : x \in g(A), A \in X\}$ *is a consistent set of expressions.*

Based on definition of set of expressions 6.11, conclusion 6.12, definition of inconsistent set of expressions 6.14 and conclusion 6.15 we get the following fact.

Proposition 6.16. *Set of expressions* Ex *meets the conditions of general definition of tableau expressions 5.15.*

Due to the above fact, set Ex will be denoted as $\mathsf{Te}_{\mathbf{MTL}}$, while its elements will be called *expressions* or *tableau expressions*. In turn, the inconsistent sets of expressions will be called *tableau inconsistent* (*t-inconsistent*), while the consistent sets of expressions will be called *tableau consistent* (*t-consistent*).

Now, we define the function selecting indices.

Definition 6.17 (Function selecting indices)**.** Let $* : \mathsf{Te}_{\mathbf{MTL}} \longrightarrow \mathbb{N}$ be such function that for any name letter $P \in \mathsf{Ln}$, any $i \in \mathbb{N}$, any $\bullet \in \{\Box, \Diamond\}$ and any formula $A \in \mathsf{For}_{\mathbf{MTL}}$:

1. $*(A^i) = i$
2. $*(P_{+i}) = i$
3. $*(P_{-i}) = i$
4. $*(P^{\bullet}_{+i}) = i$
5. $*(P^{\bullet}_{-i}) = i$.

Denotation 6.18. Let $x \in \mathsf{Te}_{\mathbf{MTL}}$ and $*(x) = i$. We adopt denotation x^i.

Now, we will define binary relation ≡ specified on the Cartesian product $P(\mathsf{Te}_{\mathbf{MTL}}) \times P(\mathsf{Te}_{\mathbf{MTL}})$ that will correspond to the general definition of similarity of sets of expressions (definition 5.23).

Definition 6.19. Let $X, Y \subseteq \mathsf{Te}_{\mathbf{MTL}}$. We define relation ≡ with condition: $X \equiv Y$ iff there exists bijection $h : *(X) \longrightarrow *(Y)$ such that for any expression $x^i \in \mathsf{Te}_{\mathbf{MTL}}$: $x^i \in X$ iff $x^{h(i)} \in Y$.

From definition 6.19 results the following conclusion.

Corollary 6.20. *Let $X, Y \subseteq \mathsf{Te}_{MTL}$. If $X \equiv Y$, then:*

- *X is t-consistent iff Y is t-consistent*
- *sets X and Y have the same cardinality.*

By conclusion 6.20 and definition of relation ≡ 6.19, we claim that relation ≡ meets the conditions of general definition of similarity of sets of expressions 5.23.

Proposition 6.21. *Relation ≡ is the relation of similarity of sets in accordance with the general definition of similarity of sets of expressions 5.23.*

We will now define the concept that establishes the relation between the models and sets of expressions.

Definition 6.22. Let X be a set of expressions, while $\mathfrak{M}_{\mathbf{MTL}} = \langle D, d^{\square}, d, d^{\diamond} \rangle$ be a model. We shall state that $\mathfrak{M}_{\mathbf{MTL}}$ is *appropriate* for set X iff the below conditions are met:

1. $\mathfrak{M}_{\mathbf{MTL}} \vDash X \cap \mathsf{For}_{\mathbf{MTL}}$
2. there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that for each name letter $P \in \mathsf{Ln}$, each $i \in \mathbb{N}$ and for any $\bullet \in \{\square, \diamond\}$:
   a. if $P_{+i} \in X$, then $\gamma(i) \in d(P)$
   b. if $P_{-i} \in X$, then $\gamma(j) \notin d(P)$
   c. if $P^{\bullet}_{+i} \in X$, then $\gamma(i) \in d^{\bullet}(P)$
   d. if $P^{\bullet}_{-i} \in X$, then $\gamma(j) \notin d^{\bullet}(P)$.

From definition of inconsistent set of expressions 6.14 and definition of model appropriate for the set of expressions 6.22 follows a condition concerning the relationship between the inconsistent sets of expressions and the appropriateness of models.

Corollary 6.23. *For any $X \subseteq \mathsf{Te}_{MTL}$, if X is t-inconsistent, then there exists no model $\mathfrak{M}_{MTL}$ appropriate for X.*

Next, note that relation $\vDash$ for **MTL** meets the conditions of relation $\Vdash$ (definition 5.25). Thus, by conclusion 6.23 and definition of appropriate model 6.22 we have the fact.

Proposition 6.24. *The notion of model appropriate for set of expressions meets the general conditions of interpretation appropriate for the set of expressions described in definition 5.28.*

Thus, we have demonstrated that the presented concepts for the tableau system for **MTL** are special cases of the general concepts described in the previous chapter.

6.2.4 Rules for the tableau system for logic MTL

We can proceed to the rules. On each rule, we conventionally assume that each of its input sets is t-consistent and that each input set is basically contained in the output set.

Definition 6.25 (Tableau rules for **MTL**)**.** *Tableau rules for system* **MTL** are the following rules:

Classical rules

$$R\mathbf{a}_+ : \frac{X \cup \{P\mathbf{a}Q, P_{+j}\}}{X \cup \{P\mathbf{a}Q, P_{+j}, Q_{+j}\}}$$

$$R\mathbf{e}_- : \frac{X \cup \{P\mathbf{e}Q, P_{+j}\}}{X \cup \{P\mathbf{e}Q, P_{+j}, Q_{-j}\}}$$

$$R\mathbf{i} : \frac{X \cup \{P\mathbf{i}Q\}}{X \cup \{P\mathbf{i}Q, P_{+j}, Q_{+j}\}}, \text{ where:}$$

1. $j \notin *(X \setminus \mathsf{For}_{\mathbf{MTL}})$
2. for any $k \in \mathbb{N}$, $\{P_{+k}, Q_{+k}\} \nsubseteq X$.

$$R\mathbf{o} : \frac{X \cup \{P\mathbf{o}Q\}}{X \cup \{P\mathbf{o}Q, P_{+j}, Q_{-j}\}}, \text{ where:}$$

1. $j \notin *(X \setminus \mathsf{For}_{\mathbf{MTL}})$
2. for any $k \in \mathbb{N}$, $\{P_{+k}, Q_{-k}\} \nsubseteq X$.

Rules for $\Diamond$

$$R\mathbf{a}_+^{\Diamond} : \frac{X \cup \{P\mathbf{a}^{\Diamond}Q, P_{+j}\}}{X \cup \{P\mathbf{a}^{\Diamond}Q, P_{+j}, Q_{+j}^{\Diamond}\}}$$

$$R\mathbf{e}^{\Diamond}_{-} : \frac{X \cup \{P\mathbf{e}^{\Diamond}Q, P_{+j}\}}{X \cup \{P\mathbf{e}^{\Diamond}Q, P_{+j}, Q^{\Box}_{-j}\}}$$

$$R\mathbf{i}^{\Diamond} : \frac{X \cup \{P\mathbf{i}^{\Diamond}Q\}}{X \cup \{P\mathbf{i}^{\Diamond}Q, P_{+j}, Q^{\Diamond}_{+j}\}}\text{, where:}$$

1. $j \notin *(X \setminus \mathsf{For}_{\mathbf{MTL}})$
2. for any $k \in \mathbb{N}$, $\{P_{+k}, Q^{\Diamond}_{+k}\} \nsubseteq X$.

$$R\mathbf{o}^{\Diamond} : \frac{X \cup \{P\mathbf{o}^{\Diamond}Q\}}{X \cup \{P\mathbf{o}^{\Diamond}Q, P_{+j}, Q^{\Box}_{-j}\}}\text{, where:}$$

1. $j \notin *(X \setminus \mathsf{For}_{\mathbf{MTL}})$
2. for any $k \in \mathbb{N}$, $\{P_{+k}, Q^{\Box}_{-k}\} \nsubseteq X$.

Rules for □

$$R\mathbf{a}^{\Box}_{+} : \frac{X \cup \{P\mathbf{a}^{\Box}Q, P_{+j}\}}{X \cup \{P\mathbf{a}^{\Box}Q, P_{+j}, Q^{\Box}_{+j}\}}$$

$$R\mathbf{e}^{\Box}_{-} : \frac{X \cup \{P\mathbf{e}^{\Box}Q, P_{+j}\}}{X \cup \{P\mathbf{e}^{\Box}Q, P_{+j}, Q^{\Diamond}_{-j}\}}$$

$$R\mathbf{i}^{\Box} : \frac{X \cup \{P\mathbf{i}^{\Box}Q\}}{X \cup \{P\mathbf{i}^{\Box}Q, P_{+j}, Q^{\Box}_{+j}\}}\text{, where:}$$

1. $j \notin *(X \setminus \mathsf{For}_{\mathbf{MTL}})$
2. for any $k \in \mathbb{N}$, $\{P_{+k}, Q^{\Box}_{+k}\} \nsubseteq X$.

$$R\mathbf{o}^{\Box} : \frac{X \cup \{P\mathbf{o}^{\Box}Q\}}{X \cup \{P\mathbf{o}^{\Box}Q, P_{+j}, Q^{\Diamond}_{-j}\}}\text{, where:}$$

1. $j \notin *(X \setminus \mathsf{For}_{\mathbf{MTL}})$
2. for any $k \in \mathbb{N}$, $\{P_{+k}, Q^{\Diamond}_{-k}\} \nsubseteq X$.

Bridging rules

$$R^{\Box}_{+} : \frac{X \cup \{P^{\Box}_{+j}\}}{X \cup \{P^{\Box}_{+j}, P_{+j}\}}$$

$$R_{+} : \frac{X \cup \{P_{+j}\}}{X \cup \{P_{+j}, P^{\Diamond}_{+j}\}}$$

$$R^{\Diamond}_{-} : \frac{X \cup \{P^{\Diamond}_{-j}\}}{X \cup \{P^{\Diamond}_{-j}, P_{-j}\}}$$

$$R_{-} : \frac{X \cup \{P_{-j}\}}{X \cup \{P_{-j}, P^{\Box}_{-j}\}}$$

Set of rules for **MTL** will be denoted as $\mathbf{R_{MTL}}$.

Note that set $\mathbf{R_{MTL}}$ meets both the general conditions of rule (definition 5.29), and the general conditions of set of tableau rules 5.33 — the cores of each rule from $\mathbf{R_{MTL}}$ (definition 6.25) are any ordered pairs in which set X is empty.

6.2.5 Branches and tableaux for MTL

We accept all general definitions from the previous chapter:

- branch 5.35
- closed/open branch 5.39
- maximal branch 5.42
- tableau 5.48
- complete tableau 5.50
- closed/open tableau 5.51
- branch consequence 5.46

assuming that these concepts are dependent on set of tableau rules $\mathbf{R_{MTL}}$.

6.2.6 Theorem on the completeness of the tableau system for MTL

In order to demonstrate that for the tableau system for **MTL** the theorem on completeness holds, we must show that the assumptions of general theorem 5.62 are met. So, we must demonstrate that:

- set of models for language of **MTL** is good for set of tableau rules $\mathbf{R_{MTL}}$ (according to general definition 5.55)
- set of tableau rules $\mathbf{R_{MTL}}$ is good for model for language of **MTL** (according to general definition 5.59).

Let us first define the concept of model defined by a branch.

Definition 6.26 (Model generated by branch)**.** Let ϕ be any branch. We define the following function $AT(\phi) = \bigcup \phi \cap (\mathsf{Te}_{\mathbf{MTL}} \setminus \mathsf{For}_{\mathbf{MTL}})$.

We shall state that model $\mathfrak{M}_{\mathbf{MTL}} = \langle D, d^{\Box}, d, d^{\Diamond} \rangle$ is *generated by branch* ϕ iff:

1. $D = \{x \in \mathbb{N} : x \in *(At(\phi))\}$
2. for any name letter $P \in \mathsf{Ln}$:

a. $x \in d(P)$ iff $P_{+x} \in At(\phi)$
b. for any $\bullet \in \{\Box, \Diamond\}$, $x \in d^{\bullet}(P)$ iff $P^{\bullet}_{+x} \in At(\phi)$.

From this definition, we get the following conclusion.

Corollary 6.27. *Let ϕ be an open and maximal branch. Then, there exists a model generated by ϕ.*

Proof. Take any open and maximal branch ϕ. From definition of open branch 5.39 and definition of model generated by branch 6.26 we get ordered quadruple $\langle D, d^{\Box}, d, d^{\Diamond} \rangle$.

We must investigate if for any name letter $P \in \mathsf{Ln}$ occurs $d^{\Box}(P) \subseteq d(P) \subseteq d^{\Diamond}(P)$. Since branch ϕ is maximal and open, so by virtue of bridging rules from definition of tableau rules $\mathbf{R_{MTL}}$ (definition 6.25), for any $i \in \mathbb{N}$ and for any name letter $P \in \mathsf{Ln}$:

- if $i \in d^{\Box}(P)$, then $i \in d(P)$, due to rule $R^{\Box}_{+}$
- if $i \in d(P)$, then $i \in d^{\Diamond}(P)$, due to rule R_{+}.

Thus, from definition of model 6.3 we get a conclusion that $\langle D, d^{\Box}, d, d^{\Diamond} \rangle$ is a model. □

As we can see, the general definition of branch generating model 5.53 gains content in the context of the tableau system of **MTL**. We might define a function assigning models to open and maximal branches, but we will refrain from doing so and instead directly use conclusion 6.27. While on model $\mathfrak{M}_{\mathbf{MTL}}$ generated by branch ϕ we shall state that branch ϕ *generates* model $\mathfrak{M}_{\mathbf{MTL}}$.

We will now show that set of models of **MTL** is good for tableau rules $\mathbf{R_{MTL}}$ (according to general definition 5.55). The following lemma will be useful for that.

Lemma 6.28. *Let ϕ be an open and maximal branch. Let $X^i \subseteq \bigcup \phi$, for some $X \subseteq \mathsf{For}_{\mathbf{MTL}}$ and $i \in \mathbb{N}$. Then branch ϕ generates such model $\mathfrak{M}_{\mathbf{MTL}}$ that $\mathfrak{M}_{\mathbf{MTL}} \vDash X$.*

Proof. Take any open and maximal branch ϕ. Take any set $X^i \subseteq \bigcup \phi$, for some $X \subseteq \mathsf{For}_{\mathbf{MTL}}$ and $i \in \mathbb{N}$. Note that in the case of **MTL** set X^i is simply a set of formulas (remark 6.13).

Since branch ϕ is open and maximal, so by virtue of the previous conclusion 6.27 there exists model $\mathfrak{M}_{\mathbf{MTL}} = \langle d^{\Box}, d, d^{\Diamond} \rangle$ generated by ϕ.

We will now show that for any formula A contained in $\bigcup \phi$, it is the case that $\mathfrak{M}_{\mathbf{MTL}} \vDash A$, i.e. $\mathfrak{M}_{\mathbf{MTL}} \vDash \bigcup \phi \cap \mathsf{For}_{\mathbf{MTL}}$. This implies that $\mathfrak{M}_{\mathbf{MTL}} \vDash X$.

The proof will be carried out by consideration of all the possible cases of construction of formula A. Assume that $A \in \bigcup \phi$. By definition of formula, for some name letters P, $Q \in \mathsf{Ln}$, there must occur one of the following cases.

1. $A = P\mathbf{a}Q$. Take any object $i \in D$ such that $i \in d(P)$. By definition of generated model 6.26, set $\bigcup\phi$ contains tableau expression P_{+i}. Since ϕ is a maximal and open branch, so by virtue of tableau rule $R\mathbf{a}_{+}$, $\bigcup\phi$ also contains tableau expression Q_{+i}. By definition of model generated 6.26, $i \in d(Q)$. Hence, $d(P) \subseteq d(Q)$, and by definition of truth in model 6.5, we thus get that $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{a}Q$. In turn, if there exists no such $i \in D$ that $i \in d(P)$, then $\varnothing = d(P) \subseteq d(Q)$, so by definition of truth in model 6.5, we get $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{a}Q$.
2. $A = P\mathbf{i}Q$. Since ϕ is a maximal and open branch, so by tableau rule $R\mathbf{i}$, set $\bigcup\phi$ also contains tableau expressions P_{+i}, Q_{+i}, for some $i \in \mathbb{N}$. By definition of model generated 6.26, $i \in d(P)$ and $i \in d(Q)$. Since $d(P) \cap d(Q) \neq \varnothing$, so by definition of truth in model 6.5, we get that $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{i}Q$.
3. $A = P\mathbf{e}Q$. Take any object $i \in D$ such that $i \in d(P)$. By definition of model generated 6.26, set $\bigcup\phi$ contains tableau expression P_{+i}. Since ϕ is a maximal and open branch, so by virtue of tableau rule $R\mathbf{e}_{-}$, $\bigcup\phi$ also contains tableau expression Q_{-i}. Since branch ϕ is open, so expression $Q_{+i} \notin \bigcup\phi$, and consequently, by definition of model generated 6.26, $i \notin d(Q)$. Thus, $d(P) \cap d(Q) = \varnothing$ and by definition of truth in model 6.5, we get $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{e}Q$. In turn, if there exists no object $i \in D$ such that $i \in d(P)$, then $d(P) \cap d(Q) = \varnothing$, so by definition of truth in model 6.5, we get that $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{e}Q$.
4. $A = P\mathbf{o}Q$. Since ϕ is a maximal and open branch, so by virtue of tableau rule $R\mathbf{o}$, set $\bigcup\phi$ also contains tableau expressions P_{+i}, Q_{-i}, for some $i \in \mathbb{N}$. By definition of model generated 6.26, $i \in d(P)$ and — since branch ϕ is open and, consequently, expression $Q_{+i} \notin \bigcup\phi$ — $i \notin d(Q)$, so $d(P) \nsubseteq d(Q)$. Thus, by definition of truth in model 6.5, we get $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{o}Q$.

The remaining eight cases of the possible construction of formula A will be reduced to four cases as both rules that we apply to them and conditions of truth that define their truth are analogous. So, take $\bullet \in \{\Box, \Diamond\}$. We have four cases.

1. $A = P\mathbf{a}^{\bullet}Q$. Take any object $i \in D$ such that $i \in d(P)$. By definition of model generated 6.26, set $\bigcup\phi$ contains tableau expression P_{+i}. Since ϕ is a maximal and open branch, so by virtue of tableau rule $R\mathbf{a}^{\bullet}_{+}$, $\bigcup\phi$ also contains tableau expression $Q^{\bullet}_{+i}$. By definition of model generated 6.26, $i \in d^{\bullet}(Q)$. Hence, $d(P) \subseteq d^{\bullet}(Q)$, and by definition of truth in model 6.5, we thus get that $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{a}^{\bullet}Q$. In turn, if there exists no such $i \in D$ that $i \in d(P)$, then $\varnothing = d(P) \subseteq d^{\bullet}(Q)$, so by definition of truth in model 6.5, we get $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{a}^{\bullet}Q$.

2. $A = P\mathbf{i}^\bullet Q$. Since ϕ is a maximal and open branch, so by virtue of tableau rule $R\mathbf{i}^\bullet$, set $\bigcup\phi$ also contains tableau expressions P_{+i}, $Q^\bullet_{+i}$, for some $i \in \mathbb{N}$. By definition of model generated 6.26, $i \in d(P)$ and $i \in d^\bullet(Q)$. Since $d(P) \cap d^\bullet(Q) \neq \varnothing$, so by definition of truth in model 6.5, we get that $\mathfrak{M}_{\mathbf{MTL}} \models P\mathbf{i}^\bullet Q$.
3. $A = P\mathbf{e}^\bullet Q$. Take any object $i \in D$ such that $i \in d(P)$. By definition of model generated 6.26, set $\bigcup\phi$ contains tableau expression P_{+i}. Since ϕ is a maximal and open branch, so by virtue of tableau rule $R\mathbf{e}^\bullet_-$, $\bigcup\phi$ also contains tableau expression $Q^{\bullet'}_{-i}$. Since branch ϕ is open, so expression $Q^{\bullet'}_{+i} \notin \bigcup\phi$, and consequently, by definition of model generated 6.26, $i \notin d(Q)^{\bullet'}$. Thus, $d(P) \cap d^{\bullet'}(Q) = \varnothing$ and by definition of truth in model 6.5, we get $\mathfrak{M}_{\mathbf{MTL}} \models P\mathbf{e}Q$. In turn, if there exists no object $i \in D$ such that $i \in d(P)$, then $d(P) \cap d^{\bullet'}(Q) = \varnothing$, so by definition of truth in model 6.5, we get that $\mathfrak{M}_{\mathbf{MTL}} \models P\mathbf{e}^\bullet Q$.
4. $A = P\mathbf{o}^\bullet Q$. Since ϕ is a maximal and open branch, so by virtue of tableau rule $R\mathbf{o}^\bullet$, set $\bigcup\phi$ also contains tableau expressions P_{+i}, $Q^{\bullet'}_{-i}$, for some $i \in \mathbb{N}$. By definition of model generated 6.26, $i \in d(P)$ and — since branch ϕ is open and, consequently, expression $Q^{\bullet'}_{+i} \notin \bigcup\phi$ — $i \notin d^{\bullet'}(Q)$, so $d(P) \nsubseteq d^{\bullet'}(Q)$. Thus, by definition of truth in model 6.5, we get $\mathfrak{M}_{\mathbf{MTL}} \models P\mathbf{o}^\bullet Q$. □

From lemma 6.28 and definition 5.55 applied to the set of models for the language of **MTL** we get a conclusion.

Corollary 6.29. *Set of models for language of* **MTL** *is good for set of tableau rules* $\boldsymbol{R}_{\boldsymbol{MTL}}$.

We will now proceed to the demonstration of an opposite dependence between rules and models.

Lemma 6.30. *Let $\mathfrak{M}_{MTL}$ be any model, X, $Y \subseteq \mathsf{Te}_{MTL}$, and let $R \in \mathbf{R}_{MTL}$. Then, if $\langle X, Y\rangle \in R$ and $\mathfrak{M}_{MTL}$ is appropriate for set of expressions X, then $\mathfrak{M}_{MTL}$ is appropriate for Y.*

Proof. In the proof, we will make use of definition of model appropriate for the set of expressions 6.22. Let $\mathfrak{M}_{\mathbf{MTL}} = \langle D, d^\square, d, d^\diamond\rangle$ be any model and X, $Y \subseteq \mathsf{Te}_{\mathbf{MTL}}$. We will consider all cases of rules $R \in \mathbf{R}_{\mathbf{MTL}}$, assuming that $\langle X, Y\rangle \in R$ and $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions X, and showing that then $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for Y.

We have four cases of rules for the classical functors.

1. Let $R = R\mathbf{a}_+$, then $\langle X, Y\rangle = \langle Z \cup \{P\mathbf{a}Q, P_{+i}\}, Z \cup \{P\mathbf{a}Q, P_{+i}, Q_{+i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{MTL}}$, $P, Q \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions X,

so by definition 6.22, $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{a}Q$ and there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that for each name letter $S \in \mathsf{Ln}$ and each $j \in \mathbb{N}$: if $S_{+j} \in X$, then $\gamma(j) \in d(S)$ and if $S_{-j} \in X$, then $\gamma(j) \notin d(S)$; due to the fact that $P_{+i} \in X$, also $\gamma(i) \in d(P)$, while since $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{a}Q$, hence by definition of truth in model 6.5, $\gamma(i) \in d(Q)$, since $d(P) \subseteq d(Q)$; consequently, by definition of model appropriate for the set of expressions 6.22, model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions $Y = Z \cup \{P\mathbf{a}Q, P_{+i}, Q_{+i}\}$.

2. Let $R = R\mathbf{i}$, then $\langle X, Y\rangle = \langle Z \cup \{P\mathbf{i}Q\}, Z \cup \{P\mathbf{i}Q, P_{+i}, Q_{+i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{MTL}}$, $P, Q \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions X, so by definition 6.22, $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{i}Q$ and there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that for each name letter $S \in \mathsf{Ln}$ and each $j \in \mathbb{N}$: if $S_{+j} \in X$, then $\gamma(j) \in d(S)$ and if $S_{-j} \in X$, then $\gamma(j) \notin d(S)$; however, rule $R\mathbf{i}$ enriches set X with expressions P_{+i}, Q_{+i} and index i is new, it has not occurred in any expression from set X, while since $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{i}Q$, so by definition of truth in model 6.5, in the domain there exists certain object x such that $x \in d(P) \cap d(Q)$; we define function $\gamma' : \mathbb{N} \longrightarrow D$ such that for any $k \in \mathbb{N}$, if $k \neq i$, then $\gamma'(k) = \gamma(k)$ and $\gamma'(i) = x$, consequently, by definition of model appropriate for the set of expressions 6.22, model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions $Y = Z \cup \{P\mathbf{i}Q, P_{+i}, Q_{+i}\}$.
3. Let $R = R\mathbf{e}_{-}$, then $\langle X, Y\rangle = \langle Z \cup \{P\mathbf{e}Q, P_{+i}\}, Z \cup \{P\mathbf{e}Q, P_{+i}, Q_{-i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{MTL}}$, $P, Q \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions X, so by definition 6.22, $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{e}Q$ and there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that for each name letter $S \in \mathsf{Ln}$ and each $j \in \mathbb{N}$: if $S_{+j} \in X$, then $\gamma(j) \in d(S)$ and if $S_{-j} \in X$, then $\gamma(j) \notin d(S)$; due to the fact that $P_{+i} \in X$, also $\gamma(i) \in d(P)$, while since $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{e}Q$, hence by definition of truth in model 6.5 $\gamma(i) \notin d(Q)$, since $d(P) \cap d(Q) = \varnothing$; consequently, by definition of model appropriate for the set of expressions 6.22, model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions $Y = Z \cup \{P\mathbf{e}Q, P_{+i}, Q_{-i}\}$.
4. Let $R = R\mathbf{o}$, then $\langle X, Y\rangle = \langle Z \cup \{P\mathbf{o}Q\}, Z \cup \{P\mathbf{o}Q, P_{+i}, Q_{-i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{MTL}}$, $P, Q \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions X, so by definition 6.22, $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{o}Q$ and there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that for each name letter $S \in \mathsf{Ln}$ and each $j \in \mathbb{N}$: if $S_{+j} \in X$, then $\gamma(j) \in d(S)$ and if $S_{-j} \in X$, then $\gamma(j) \notin d(S)$; however, rule $R\mathbf{o}$ enriches set X with expressions P_{+i}, Q_{-i} and index i is new, it has not occurred in any expression from set X, while since $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{o}Q$, so by definition of truth in model 6.5, in the domain there exists certain object x such that $x \in d(P)$, but $x \notin d(Q)$; we define function $\gamma' : \mathbb{N} \longrightarrow D$ such that for any $k \in \mathbb{N}$, if $k \neq i$, then $\gamma'(k) = \gamma(k)$ and $\gamma'(i) = x$, consequently, by definition of model appropriate for the set of expressions 6.22, model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions $Y = Z \cup \{P\mathbf{o}Q, P_{+i}, Q_{-i}\}$.

We have eight rules for formulas with modal functors. However, we will restrict our considerations to four cases as they are analogous in terms of application. So, take $\bullet \in \{\Box, \Diamond\}$.

1. Let $R = R\mathbf{a}^{\bullet}_{+}$, then $\langle X, Y\rangle = \langle Z \cup \{P\mathbf{a}^{\bullet}Q, P_{+i}\}, Z \cup \{P\mathbf{a}^{\bullet}Q, P_{+i}, Q^{\bullet}_{+i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{MTL}}$, P, $Q \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions X, so by definition 6.22, $\mathfrak{M}_{\mathbf{MTL}} \models P\mathbf{a}^{\bullet}Q$ and there exists function $\gamma : \mathbb{N} \longrightarrow D$ i.a. such that for each name letter $S \in \mathsf{Ln}$ and each $j \in \mathbb{N}$: if $S_{+j} \in X$, then $\gamma(j) \in d(S)$ and if $S^{\bullet}_{+j} \in X$, then $\gamma(j) \in d^{\bullet}(S)$; due to the fact that $P_{+i} \in X$, also $\gamma(i) \in d(P)$, while since $\mathfrak{M}_{\mathbf{MTL}} \models P\mathbf{a}^{\bullet}Q$, hence by definition of truth in model 6.5, $\gamma(i) \in d^{\bullet}(Q)$, since $d(P) \subseteq d^{\bullet}(Q)$; consequently, by definition of model appropriate for the set of expressions 6.22, model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions $Y = Z \cup \{P\mathbf{a}^{\bullet}Q, P_{+i}, Q^{\bullet}_{+i}\}$.
2. Let $R = R\mathbf{i}^{\bullet}$, then $\langle X, Y\rangle = \langle Z \cup \{P\mathbf{i}^{\bullet}Q\}, Z \cup \{P\mathbf{i}^{\bullet}Q, P_{+i}, Q^{\bullet}_{+i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{MTL}}$, P, $Q \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions X, so by definition 6.22, $\mathfrak{M}_{\mathbf{MTL}} \models P\mathbf{i}^{\bullet}Q$ and there exists function $\gamma : \mathbb{N} \longrightarrow D$ i.a. such that for each name letter $S \in \mathsf{Ln}$ and each $j \in \mathbb{N}$: if $S_{+j} \in X$, then $\gamma(j) \in d(S)$ and if $S^{\bullet}_{+j} \in X$, then $\gamma(j) \in d^{\bullet}(S)$; however, rule $R\mathbf{i}^{\bullet}$ enriches set X with expressions $P_{+i}, Q^{\bullet}_{+i}$ and index i is new, it has not occurred in any of expressions from set X, while since $\mathfrak{M}_{\mathbf{MTL}} \models P\mathbf{i}^{\bullet}Q$, so by definition of truth in model 6.5, in the domain there exists certain object x such that $x \in d(P) \cap d^{\bullet}(Q)$; we define function $\gamma' : \mathbb{N} \longrightarrow D$ such that for any $k \in \mathbb{N}$, if $k \neq i$, then $\gamma'(k) = \gamma(k)$ and $\gamma'(i) = x$, consequently, by definition of model appropriate for the set of expressions 6.22, model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions $Y = Z \cup \{P\mathbf{i}^{\bullet}Q, P_{+i}, Q^{\bullet}_{+i}\}$.
3. Let $R = R\mathbf{e}^{\bullet}_{-}$, then $\langle X, Y\rangle = \langle Z \cup \{P\mathbf{e}^{\bullet}Q, P_{+i}\}, Z \cup \{P\mathbf{e}^{\bullet}Q, P_{+i}, Q^{\bullet'}_{-i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{MTL}}$, P, $Q \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions X, so by definition 6.22, $\mathfrak{M}_{\mathbf{MTL}} \models P\mathbf{e}^{\bullet}Q$ and there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that i.a. for each name letter $S \in \mathsf{Ln}$ and each $j \in \mathbb{N}$: if $S_{+j} \in X$, then $\gamma(j) \in d(S)$ and if $S^{\bullet'}_{-j} \in X$, then $\gamma(j) \notin d^{\bullet'}(S)$; due to the fact that $P_{+i} \in X$, also $\gamma(i) \in d(P)$, while since $\mathfrak{M}_{\mathbf{MTL}} \models P\mathbf{e}^{\bullet}Q$, hence by definition of truth in model 6.5 $\gamma(i) \notin d^{\bullet'}(Q)$, since $d(P) \cap d^{\bullet'}(Q) = \varnothing$; consequently, by definition of model appropriate for the set of expressions 6.22, model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions $Y = Z \cup \{P\mathbf{e}^{\bullet}Q, P_{+i}, Q^{\bullet'}_{-i}\}$.
4. Let $R = R\mathbf{o}^{\bullet}$, then $\langle X, Y\rangle = \langle Z \cup \{P\mathbf{o}^{\bullet}Q\}, Z \cup \{P\mathbf{o}^{\bullet}Q, P_{+i}, Q^{\bullet'}_{-i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{MTL}}$, P, $Q \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions X, so by definition 6.22, $\mathfrak{M}_{\mathbf{MTL}} \models P\mathbf{o}^{\bullet}Q$ and there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that for each name letter $S \in \mathsf{Ln}$ and each $j \in \mathbb{N}$: if $S_{+j} \in X$, then $\gamma(j) \in d(S)$ and if $S^{\bullet'}_{-j} \in X$, then $\gamma(j) \notin d^{\bullet'}(S)$; however, rule $R\mathbf{o}^{\bullet}$ enriches set X with expressions

$P_{+i}, Q^{\bullet'}_{-i}$ and index i is new, it has not occurred in any expression from set X, while since $\mathfrak{M}_{\mathbf{MTL}} \models P\mathbf{o}^{\bullet}Q$, so by definition of truth in model 6.5, in the domain there exists certain object x such that $x \in d(P)$, but $x \notin d^{\bullet'}(Q)$; we define function $\gamma' : \mathbb{N} \longrightarrow D$ such that for any $k \in \mathbb{N}$, if $k \neq i$, then $\gamma'(k) = \gamma(k)$ and $\gamma'(i) = x$, consequently, by definition of model appropriate for the set of expressions 6.22, model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions $Y = Z \cup \{P\mathbf{o}^{\bullet}Q, P_{+i}, Q^{\bullet'}_{-i}\}$.

We have four cases for the bridging rules.

1. Let $R = R^{\Box}_{+}$, then $\langle X, Y\rangle = \langle Z \cup \{P^{\Box}_{+i}\}, Z \cup \{P^{\Box}_{+i}, P_{+i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{MTL}}$, $P \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions X, so by definition 6.22, there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that $\gamma(i) \in d^{\Box}(P)$; furthermore $d^{\Box}(P) \subseteq d(P)$, by definition of model 6.3, so $\gamma(i) \in d(P)$; thus, by definition of model appropriate for the set of expressions 6.22, model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions $Y = Z \cup \{P^{\Box}_{+i}, P_{+i}\}$.
2. Let $R = R_{+}$, then $\langle X, Y\rangle = \langle Z \cup \{P_{+i}\}, Z \cup \{P_{+i}, P^{\Diamond}_{+i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{MTL}}$, $P \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions X, so by definition 6.22, there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that $\gamma(i) \in d(P)$; furthermore $d(P) \subseteq d^{\Diamond}(P)$, by definition of model 6.3, so $\gamma(i) \in d^{\Diamond}(P)$; thus, by definition of model appropriate for the set of expressions 6.22, model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions $Y = Z \cup \{P_{+i}, P^{\Diamond}_{+i}\}$.
3. Let $R = R^{\Diamond}_{-}$, then $\langle X, Y\rangle = \langle Z \cup \{P^{\Diamond}_{-i}\}, Z \cup \{P^{\Diamond}_{-i}, P_{-i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{MTL}}$, $P \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions X, so by definition 6.22, there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that $\gamma(i) \notin d^{\Diamond}(P)$; furthermore $d(P) \subseteq d^{\Diamond}(P)$, by definition of model 6.3, so $\gamma(i) \notin d(P)$; thus, by definition of model appropriate for the set of expressions 6.22, model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions $Y = Z \cup \{P^{\Diamond}_{-i}, P_{-i}\}$.
4. Let $R = R_{-}$, then $\langle X, Y\rangle = \langle Z \cup \{P_{-i}\}, Z \cup \{P_{-i}, P^{\Box}_{-i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{MTL}}$, $P \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions X, so by definition 6.22, there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that $\gamma(i) \notin d(P)$; furthermore $d^{\Box}(P) \subseteq d(P)$, by definition of model 6.3, so $\gamma(i) \notin d^{\Box}(P)$; thus, by definition of model appropriate for the set of expressions 6.22, model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions $Y = Z \cup \{P_{-i}, P^{\Box}_{-i}\}$. □

From lemma 6.30 and definition 5.59 applied to set of tableau rules $\mathbf{R}_{\mathbf{MTL}}$ we get a conclusion.

Corollary 6.31. *Set of tableau rules $\mathbf{R_{MTL}}$ is good for set of models for the language of **MTL**.*

Applying both conclusions: 6.29 and 6.31, concepts defined in this chapter and general tableau theorem 5.62, from the previous chapter, we get the completeness theorem for system of **MTL**.

Theorem 6.32 (Completeness theorem for **MTL**). *For any set $X \subseteq \mathsf{For}_{\mathbf{MTL}}$ and any formula $A \in \mathsf{For}_{\mathbf{MTL}}$ the following statements are equivalent:*

- $X \vDash A$
- $X \rhd A$
- *there exists finite set $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi \rangle$*

Thus, we have shown how — with application of the general tableau concepts — we can shorten the proof of completeness theorem to the appropriate definition of specific concepts and application of the general theorem.

6.2.7 Estimation of cardinality of model for MTL

When applying the tableau methods to **TL** we received a possibility of estimation of the limitation of cardinality of models that can be countermodels for the considered inference (theorem 3.57). The situation is similar in the case of system for **MTL**. We can get an identical outcome nearly directly from the completeness theorem of the tableau system for **MTL**, carrying out a proof analogous to the one for theorem 3.57.

By *existential formula*, we mean any formula in form $P\mathbf{i}Q$, $P\mathbf{o}Q$, $P\mathbf{i}^{\bullet}Q$, $P\mathbf{o}^{\bullet}Q$, where $P, Q, \in \mathsf{Ln}$ and $\bullet \in \{\Box, \Diamond\}$.

Next, we define function $\lambda' : P(\mathsf{For}_{\mathbf{TL}}) \longrightarrow P(\mathsf{For}_{\mathbf{TL}})$ with the following condition: for any set $\Phi \in P(\mathsf{For}_{\mathbf{TL}})$, $\lambda'(\Phi) = \{x \in \Phi : x \text{ is an existential formula}\}$. So, from each set of rules, function λ' "selects" all existential formulas that belong to a given set.

Now, in turn, we shall specify function $\sigma : \{\Psi \in P(\mathsf{For}_{\mathbf{TL}}) : \Psi \text{ is a finite set}\} \longrightarrow \mathbb{N}$ with the following condition: for any finite set $\Psi \in P(\mathsf{For}_{\mathbf{TL}})$, $\sigma'(\Psi) = |\lambda'(\Psi)|$. So, function σ' "counts" the number of existential formulas that are found in any finite set of formulas.

With the use of the defined functions, we can frame the following theorem.

Theorem 6.33. *Let X be a finite set of formulas and let $A \in \mathsf{For}_{\mathbf{MTL}}$. Then, the below statements are equivalent:*

- *for any model $\mathfrak{M}_{\mathbf{MTL}} = \langle D, d^{\Box}, d, d^{\Diamond} \rangle$, if $|D| = \sigma'(X \cup \{\circ(A)\})$ and $\mathfrak{M}_{\mathbf{MTL}} \vDash X$, then $\mathfrak{M}_{\mathbf{MTL}} \vDash A$*
- $X \vDash A$.

6.3 Tableau systems for modal logics

In this subchapter, we will investigate the application of the theory of tableau systems we covered in Chapter Five to the general case. We will show how to apply the general concepts described previously to construct tableau systems for modal logics determined by models with possible worlds.

By using general concepts, we will provide the conditions whose occurrence demonstration is sufficient to get a complete tableau system for a given modal logic.[5]

6.3.1 Language and semantics

We adopt the set of formulas for modal logic defined in one of the previous chapters with definition 4.2 for logic **S5**. We will denote that set as $\mathsf{For}_{\mathbf{ML}}$.

Next, we adopt the general concept of model with possible worlds $\mathfrak{M}_{\mathbf{ML}}$, in accordance with definition of model 4.5 for logic **S5** — with any relation of accessibility. We will denote the set of all such models as **M**.

We define the concept of truth (and falsehood) of formula in model, analogously to definition 4.7.

According to definition 5.8, each model $\mathfrak{M}_{\mathbf{ML}} \in \mathbf{M}$ can be identified with interpretation of formulas as for function $f : \mathsf{For}_{\mathbf{ML}} \longrightarrow \mathsf{For}_{\mathbf{ML}}$ defined with condition $f(A) = \neg A$, for any $A \in \mathsf{For}_{\mathbf{ML}}$ it is the case that $A \in \{B \in \mathsf{For}_{\mathbf{ML}} : \mathfrak{M}_{\mathbf{ML}} \models B\}$ iff $f(A) \notin \{B \in \mathsf{For}_{\mathbf{ML}} : \mathfrak{M}_{\mathbf{ML}} \models B\}$.

Taking any subset $\mathbf{M}' \subseteq \mathbf{M}$, we conventionally define relation of semantic consequence $\models_{\mathbf{M}'}$. Based on that relation, we semantically define modal logic $\langle \mathsf{For}_{\mathbf{ML}}, \models_{\mathbf{M}'} \rangle$.

6.3.2 Tableau expressions

Now, we will proceed to the issue of expressions representing formulas and other properties in the tableau proof.

First, we define the set of expressions. Next, we will show that it meets the general conditions imposed on the set of tableau expressions (definition 5.15).

Definition 6.34 (Expressions)**.** Set of expressions Ex is a set that is the union of the below sets:

5 The issues described in this subchapter were partially presented in article [8]. However, the general tableau theorem 5.62 was not used there. In addition, some concepts were defined differently in that paper.

- $\mathsf{For}_{\mathbf{ML}} \times \mathbb{N}$
- $\{irj : i,j \in \mathbb{N}\}$
- $\{\sim irj : i,j \in \mathbb{N}\}$
- $\{i = j : i,j \in \mathbb{N}\}$
- $\{\sim i = j : i,j \in \mathbb{N}\}$.

The elements of set $\mathbb{N}$ are called *indices*.

Remark 6.35. New types of expressions appeared in the set of expressions, which in the proof correspond to the negation of relation occurrence, the identity and negation of identity. They are not needed in all constructed systems, but in some systems they will be used by the tableau rules.

We can define the following function $g : \mathsf{For}_{\mathbf{ML}} \longrightarrow P(\mathsf{Ex})$, for any $A \in \mathsf{For}_{\mathbf{ML}}$, $g(A) = \{\langle A, i\rangle : i \in \mathbb{N}\}$. Function g has important properties, for any two formulas A, B: $A \neq B$ iff $g(A) \cap g(B) = \varnothing$, and $g(A)$ is a countable subset of set Ex.

Note, furthermore, that each expression $\langle A, i\rangle$ can be identified with expression A^i, which corresponds to function g from general definition of tableau expressions 5.15.

Definition 6.36. Let $X \subseteq \mathsf{Ex}$. We shall state that X is *tableau inonsistent* (for short: t-inconsistent) iff for some $A \in \mathsf{For}_{\mathbf{ML}}$, $i,j \in \mathbb{N}$ at least one of the below conditions is met:

1. $\langle A, i\rangle$, $\langle \neg A, i\rangle \in X$
2. irj, $\sim irj \in X$
3. $i = j$, $\sim i = j \in X$.

We shall state that X is *tableau consistent* (for short: t-consistent) iff it is not tableau inconsistent.

From definition 6.36 results the following conclusion.

Corollary 6.37. *Let $X \subseteq \mathsf{For}_{\mathbf{ML}}$ and $\mathfrak{M}_{\mathbf{ML}} \in \boldsymbol{M}$. If $\mathfrak{M}_{\mathbf{ML}} \models X$, then $\{x^i : x^i \in g(A), A \in X\}$ is a t-consistent set.*

By virtue of definition of expressions 6.34, definition 6.36, existence of function g, conclusion 6.37 and general definition of tableau expressions 5.15, we can assume that set of expressions Ex is a set of tableau expressions. The set will be denoted as $\mathsf{Te}_{\mathbf{ML}}$.

Now, we will define more auxiliary concepts.

Definition 6.38 (Function selecting indices)**.** Function $* : P(\mathsf{Te}_{\mathbf{ML}}) \longrightarrow P(\mathbb{N})$ is a *function selecting indices* iff for any $A \in \mathsf{For}_{\mathbf{ML}}$, $i,j \in \mathbb{N}$ and $X \subseteq \mathsf{Te}_{\mathbf{ML}}$ the below conditions are met:

- $*(\{\langle A,i\rangle\}) = \{i\}$
- $*(\{irj\}) = \{i,j\}$
- $*(\{\sim irj\}) = \{i,j\}$
- $*(\{i = j\}) = \{i,j\}$
- $*(\{\sim i = j\}) = \{i,j\}$
- if $|X| > 1$, then $*(X) = \bigcup\{*(\{x\}) : x \in X\}$.

For any subset Y of set of expressions $\mathsf{Te}_{\mathbf{ML}}$ function $*$ selects all indices present in Y.

Now, we will define binary relation $\equiv$ specified on Cartesian product $P(\mathsf{Te}_{\mathbf{ML}}) \times P(\mathsf{Te}_{\mathbf{ML}})$, that will correspond to the general definition of similarity of sets of expressions (5.23).

Definition 6.39. Let X, $Y \subseteq \mathsf{Te}_{\mathbf{ML}}$. We define relation $\equiv$ with condition: $X \equiv Y$ iff there exists bijection $h : *(X) \longrightarrow *(Y)$ (where $*(X)$, $*(Y)$ are sets of indices present in the expressions from X and from Y) such that for any $A \in \mathsf{For}_{\mathbf{ML}}$, $i,j \in \mathbb{N}$:

- $\langle A,i\rangle \in X$ iff $\langle A,h(i)\rangle \in Y$
- $irj \in X$ iff $h(i)rh(j) \in Y$
- $\sim irj \in X$ iff $\sim h(i)rh(j) \in Y$
- $i = j \in X$ iff $h(i) = h(j) \in Y$
- $\sim i = j \in X$ iff $\sim h(i) = h(j) \in Y$.

From definition 6.39 results the following conclusion.

Corollary 6.40. *Let X, $Y \subseteq \mathsf{Te}_{\mathbf{ML}}$. If $X \equiv Y$, then:*

- *X is t-consistent iff Y is t-consistent*
- *sets X and Y have the same cardinality.*

By virtue of conclusion 6.40 and definition of relation $\equiv$ 6.39 we claim that relation $\equiv$ meets the conditions of general definition of similarity of sets of expressions 5.23.

We will now define the concept that describe the relation between the models and sets of expressions.

Definition 6.41. Let $\mathfrak{M}_{\mathbf{ML}} = \langle W, Q, V, w\rangle$ and $X \subseteq \mathsf{Te}_{\mathbf{ML}}$. We shall state that model $\mathfrak{M}_{\mathbf{ML}}$ is *appropriate* for set of expressions X iff there exists such function $\gamma : \mathbb{N} \longrightarrow W$, that for any $A \in \mathsf{For}_{\mathbf{ML}}$, $i,j \in \mathbb{N}$ the following conditions occur:

- if $\langle A,i\rangle \in X$, then $\langle W, Q, V, \gamma(i)\rangle \models A$
- if $irj \in X$, then $\gamma(i)Q\gamma(j)$
- if $\sim irj \in X$, then it is not the case that $\gamma(i)Q\gamma(j)$

- if $i = j \in X$, then $\gamma(i)$ is identical to $\gamma(j)$
- if $\sim i = j \in X$, then $\gamma(i)$ is different from $\gamma(j)$.

From definition of tableau inconsistent set of expressions 6.36 and definition of model appropriate for set of expressions 6.41 the following conclusion results.

Corollary 6.42. *For any set of expressions $X \subseteq \mathsf{Te}_{ML}$, if X is t-inconsistent, then there exists no model $\mathfrak{M}_{ML}$ appropriate for X.*

Note that relation $\vDash$ defined by any subset $\mathbf{M}' \subseteq \mathbf{M}$ meets the conditions of relation $\Vdash$ (definition 5.25). Thus, by conclusion 6.42 and definition of appropriate model 6.41 we get a fact.

Proposition 6.43. *The notion of model appropriate for set of expressions meets the general conditions of interpretation appropriate for set of expressions described in definition 5.25.*

Thus, we have demonstrated that the presented concepts for the modal logics, determined with the semantics of possible worlds, are special cases of the general concepts described in the previous chapter.

6.3.3 Rules, branches and tableaux for modal logics

First, we will adopt the concept of tableau rule, originating by the application of concept of tableau expression $\mathsf{Te}_{\mathbf{ML}}$ and other presented concepts for modal logics to the general concept of rule 5.29 and general concept of tableau rule 5.33. Set of tableau rules for a given modal logic will be denoted as $\mathbf{R}_{\mathbf{ML}}$.

Furthermore, we adopt all general definitions from the previous chapter:

- branch 5.35
- closed/open branch 5.39
- maximal branch 5.42
- tableau 5.48
- complete tableau 5.50
- closed/open tableau 5.51
- branch consequence 5.46

assuming that these concepts are always dependent on some fixed set of tableau rules $\mathbf{R}_{\mathbf{ML}}$ for a given modal logic.

6.3.4 Generating of model

As we mentioned in the previous chapter, in remark 5.54, it is difficult to establish a general method for transition from the maximal and open branch to the generated model. For we do not know how the model is constructed and what types of expressions are used in the proof which leads to the maximal and open branch. Nonetheless, for a single logic it is possible — we did so in the case of a tableau system for **MTL**.

We can try to do the same in the case discussed, i.e. in relation to certain class of logics which in many respects are similar. The definition provided below is quite broad and on its basis we can define many types of models for modal logics determined by the semantics of possible worlds. For we have the general concept of model and the concept of set of tableau expressions $\mathsf{Te}_{\mathbf{ML}}$ which determines the range and elements used to define the model.

Definition 6.44 (Branch generating model)**.** Let $\mathbf{R}_{\mathbf{ML}}$ be a set of tableau rules and ϕ be $\mathbf{R}_{\mathbf{ML}}$-branch. Let $X = \{\langle A,k\rangle : A \in Y\} \subseteq \bigcup\phi$, for some $k \in \mathbb{N}$ and non-empty subset $Y \subseteq \mathsf{For}_{\mathbf{ML}}$. We define set $AT(\phi)$ as follows: $x \in AT(\phi)$ iff one of the below conditions occurs

- $x \in \bigcup\phi \cap (\{irj : i,j \in \mathbb{N}\} \cup \{i = j : i,j \in \mathbb{N}\})$
- $x \in \bigcup\phi \cap (\mathsf{Var} \times \mathbb{N})$.

We shall state that branch ϕ *generates* model $\mathfrak{M}_{\mathbf{ML}} = \langle W, Q, V, w\rangle$ iff

- W is a maximal subset of set $\{i : i \in *(AT(\phi))\}$ such that:
  a. for any $i,j \in \mathbb{N}$, if $i = j \in AT(\phi)$, then $i \notin W$ or $j \notin W$
  b. $k \in W$
- for any $i,j \in \mathbb{N}$
  a. $\langle i,j\rangle \in Q$ iff $irj \in AT(\phi)$ and $i,j \in W$
  b. $V(x,i) = 1$ iff $\langle x,i\rangle \in AT(\phi)$ and $i \in W$
- $w = k$.

Remark 6.45. In definition of branch generating model 6.44, the domain of model W was specified i.a. as follows: W is a maximal subset of set $\{i : i \in *(AT(\phi))\}$ such that for any $i,j \in \mathbb{N}$, if $i = j \in AT(\phi)$, then $i \notin W$ or $j \notin W$. Model generated by an open and maximal branch consists of indices included in the tableau expressions. In the event when the branch contains expression $i = j$, for some $i,j \in \mathbb{N}$, we must select one of the indices that belong to $*(\{i = j\})$, since expression $i = j$ is expected to state that it is about the same object in the domain.

Definition 6.44 obviously meets the general conditions of the definition of branch generating interpretation, since by virtue of the next conclusion, each open and maximal branch can be assigned a model.

Corollary 6.46. *Let $\mathbf{R}_{ML}$ be a set of tableau rules. Let ϕ be an open and maximal $\mathbf{R}_{ML}$-branch and let $X = \{\langle A,k\rangle : A \in Y\} \subseteq \bigcup\phi$, for some $k \in \mathbb{N}$ and non-empty subset $Y \subseteq \mathsf{For}_{ML}$. Then there exists model $\mathfrak{M}_{ML}$ such that branch ϕ generates $\mathfrak{M}_{ML}$.*

Proof. We get that conclusion from definition of open branch 5.39 applied to the modal tableau rules and definition of branch generating model 6.46. □

So, checking for a given set of tableau rules $\mathbf{R_{ML}}$ and given class of models $\mathbf{M}' \subseteq \mathbf{M}$, if set $\mathbf{M}'$ is good for set of rules $\mathbf{R_{ML}}$, does not require demonstration of the existence of model. We only have to — in accordance with definition 5.55 — demonstrate that in the generated model true are those formulas whose equivalents belonged to the branch and were used for the definition of model and that this model belongs to set $\mathbf{M}'$.

6.3.5 Completeness theorem of tableau systems for modal logics

We will now verbalize a general theorem on completeness of tableau systems for modal logics defined with the semantics of possible worlds.

Applying the general definitions of set of interpretations good for set of rules 5.55 and set of rules good for set of interpretations 5.59, the concepts defined in this chapter and general tableau theorem 5.62 we proved in the previous chapter, we obtain a general theorem on completeness for modal logics of defined with the semantics of possible worlds.

In order to frame the theorem, let us assume that for any set of models $\mathbf{M}' \subseteq \mathbf{M}$ and any set of tableau rules $\mathbf{R_{ML}}$ notation $X \vDash_{\mathbf{M}'} A$ means relation $\vDash$ defined by set of models $\mathbf{M}'$, whereas notation $X \rhd_{\mathbf{R_{ML}}} A$ means relation $\rhd$ defined by set of tableau rules $\mathbf{R_{ML}}$. Similarly, when writing $\mathbf{R_{ML}}$-tableau $\langle Y, A, \Phi \rangle$, we mean that tableau $\langle Y, A, \Phi \rangle$ and branches that belong to set of branches Φ originated merely by the application of rules from set $\mathbf{R_{ML}}$ to the expressions from set $\mathsf{Te}_{\mathbf{ML}}$.

Let us proceed to the theorem.

Theorem 6.47. *Let $\mathbf{M}' \subseteq \mathbf{M}$ be a set of models. Let $\mathbf{R_{ML}}$ be a set of tableau rules. If:*

- *set $\mathbf{M}'$ is good for set of rules $\mathbf{R_{ML}}$*
- *set $\mathbf{R_{ML}}$ is good for set of models $\mathbf{M}'$,*

then for any set $X \subseteq \mathsf{For}_{\mathbf{ML}}$ and any formula $A \in \mathsf{For}_{\mathbf{ML}}$ the following statements are equivalent:

- $X \vDash_{\mathbf{M}'} A$
- $X \rhd_{\mathbf{R_{ML}}} A$
- *there exists such finite set Y that $Y \subseteq X$ and closed $\mathbf{R_{ML}}$-tableau $\langle Y, A, \Phi \rangle$.*

Theorem 6.47 reduces the constitution of complete tableau system for modal logic defined by the semantics of possible worlds to the demonstration of two facts that form the assumptions of this theorem. So, with the established set of models $\mathbf{M}'$ we must define the set of tableau rules so $\mathbf{R_{ML}}$ as to these two facts occur. Analogously, we may start from set of tableau rules $\mathbf{R_{ML}}$ and specify subset $\mathbf{M'} \subseteq \mathbf{M}$ of all models with possible worlds so as to the theorem assumptions occur. In both cases we will get a complete tableau system.

6.4 Tableau system

In this study, we have often used term *tableau system*, even though the emphasis was on the concept of tableau and branch consequence. Ultimately, however, it is the tableau system that constitutes the tableau recognition of given logic. Therefore, we will now try to clarify the concept of tableau system.

By a tableau system we may mean pair $\langle \mathsf{For}, \rhd_{\mathbf{R}} \rangle$, where For is a set of formulas of given logic, $\mathbf{R}$ set of tableau rules, whereas $\rhd$ is a relation of branch consequence, defined based on the set of all maximal $\mathbf{R}$-branches.

Such approach of the tableau system indirectly contains all concepts we have defined when constructing relation $\rhd$. To define the relation of branch consequence, it takes i.a. the concept of tableau expressions, t-inconsistent set, concept of tableau rules, description of relationship between the formulas and tableau expressions as well as the concepts of open/closed and maximal branches.

The concept of tableau and its various variants (open/closed tableau, complete/incomplete tableau) can be, in turn, regarded as an apprehension of the method for choosing a relatively small subset of set of branches that allows to determine the occurrence of relation of branch consequence $\rhd$.

The concepts presented here enable the investigation of the relationships between different tableau systems defined by the method described, e.g. related to the questions about the dependency of sets of tableau rules or to the economics of sets system construction in general. For example, using the modification of the proof of the general completeness theorem 5.62 (for one set of tableau rules, considering the implication (a), and for another, considering the implications (b) and (c)), we can frame a theorem on the relationships between the tableau systems.

Theorem 6.48. *Let $\langle \mathsf{For}, \models_{\boldsymbol{M}} \rangle$ be a logic semantically defined with set of interpretations $\boldsymbol{M}$. Let $\langle \mathsf{For}, \rhd_{\boldsymbol{R}} \rangle$ and $\langle \mathsf{For}, \rhd_{\boldsymbol{R'}} \rangle$ be tableau systems. If:*

1. *set $\boldsymbol{M}$ is good for set of rules $\boldsymbol{R'}$*
2. *set $\boldsymbol{R}$ is good for set of interpretations $\boldsymbol{M}$*

then for any set of formulas $X \subseteq \mathsf{For}$ and any formula $A \in \mathsf{For}$, if $X \rhd_{\boldsymbol{R}} A$, then $X \rhd_{\boldsymbol{R'}} A$.

Note that sets of tableau rules **R** and **R'** can be defined on completely different sets of tableau expressions, using different auxiliary concepts, so that the rules they include can be directly incomparable. But comparing their deductive power can be done by checking the dependencies between the sets of tableau rules and the set of interpretations **M**.

6.5 Transition from the formalized tableaux to the standard tableaux

The last issue we will be dealing with is the problem of the relationship between the concepts presented in the book — the concept of branch and the concept of tableau — and the conventionally comprehended tableaux.

We discuss this problem in the part devoted to the application of the theory of tableaux, guided by the belief that standard tableaux are a practical application of abstract concepts.[6]

However, since in most cases tableaux and tableau systems are presented primarily in graphic form, our goal was to create a formalization, and thus the theory of tableaux, independent of the tableaux comprehended in this way. The theory described here is precisely an abstract approach to the tableau methods for which the standard tableaux can be considered as applications.

Take any tableau system $\langle \mathsf{For}, \triangleright_{\mathbf{R}} \rangle$, where For is a set of formulas, whereas **R** is a set of tableau rules. From general definition of tableau rules 5.33, it follows that the set of rules **R** is associated to a minimal set of tableau expressions Te which was used for the definition of rules from set **R**.

Consider any branch created by the application of rules from set **R**. In accordance with general definition of branch 5.35, that branch is a certain injective function $\phi : K \longrightarrow P(\mathsf{Te})$, where $K = \mathbb{N}$ or $K = \{1, 2, 3, \ldots, n\}$, for some $n \in \mathbb{N}$, while $P(\mathsf{Te})$ is a set of all subsets of the set of tableau expressions.

We will now define an intuitive branch that will correspond to branch ϕ. Let M be linearly ordered by relation $\leq_M$ set of points such that there exists bijection $\alpha : K \longrightarrow M$ that meets condition $\alpha(i) \leq_M \alpha(j)$, if $i \leq j$, for any $i, j \in K$.

Now, we define function $\phi' : M \longrightarrow P(\mathsf{Te})$ from the set of points into the set of all subsets of the set of tableau expressions by the following condition for any $i \in M$:

1. $\phi'(i) = \phi(1)$, if $\alpha^{-1}(i) = 1$
2. $\phi'(i) = \phi(n) \smallsetminus \phi(n-1)$, if $\alpha^{-1}(i) = n$ and $n > 1$, for any $n \in K$.

Function ϕ' specifies a linearly ordered set of points in which each point has a certain subset Te assigned. Anyway, point $\alpha(1)$ has assigned entire set $\phi(1)$ from

6 So we share the remark of Melvin Fitting who wrote in *Handbook of Tableau Methods* that the proof trees (graphs), applied as proofs in the tableau systems, make up an application of a more abstract approach to logics (p. 5, [2]).

branch ϕ — the literature usually refers to that point as *root* — while the other points have assigned sets of elements that make up the difference between the elements of branch ϕ and their immediate predecessors.

In this way, we can assign to each branch ϕ at least one branch ϕ', with some established domain — set of points M linearly ordered by relation $\leq_M$.

Example 6.49. Take set of tableau expressions $\{\langle\Diamond p,1\rangle, \langle\Diamond(p \wedge q),1\rangle, \langle\neg\neg q,1\rangle\}$. Applying rules from $\mathbf{R_{S5}}$ in the order described on left hand side, we get the following branch which will be defined as ψ.

$$
\begin{array}{ll}
R_{\Diamond} & X_1 = \{\langle\Diamond p, 1\rangle, \langle\Diamond(p \wedge q), 1\rangle, \langle\neg\neg q, 1\rangle\} \\
& \quad | \\
R_{\wedge} & X_2 = X_1 \cup \{1r2, \langle(p \wedge q), 2\rangle\} \\
& \quad | \\
R_{\neg\neg} & X_3 = X_2 \cup \{\langle p, 2\rangle, \langle q, 2\rangle\} \\
& \quad | \\
& X_4 = X_3 \cup \{\langle q, 1\rangle\}
\end{array}
$$

Transition from branch ϕ to function ϕ' will be called a *branch translation* or simply *translation*. If function $\phi' : M \longrightarrow P(\mathsf{Te})$ is a translation, then notation ϕ'_M indicates domain of ϕ'.

The result of certain translation ϕ' will be presented in the graphs below. Example 6.49 described branch ψ. Now, if we carry out the translation of branch ψ, then we can get the following graph presenting branch ψ'. This time — unlike so far — we will put the application of the rule on the right hand side and next to the points of branch that resulted from the application of given rule.

$$
\begin{array}{ll}
1.\ \{\langle\Diamond p, 1\rangle, \langle\Diamond(p \wedge q), 1\rangle, \langle\neg\neg q, 1\rangle\} & \\
\quad | & \\
2.\ \{1r2, \langle(p \wedge q), 2\rangle\} & R_{\Diamond} \\
\quad | & \\
3.\ \{\langle p, 2\rangle, \langle q, 2\rangle\} & R_{\wedge} \\
\quad | & \\
4\ \{\langle q, 1\rangle\} & R_{\neg\neg}
\end{array}
$$

And if we remove the numbering, brackets of sets and brackets from the tableau expressions, and in lieu of numbers we will insert points •, then we will get a graph of branch ψ' corresponding to the graphs usually used to present branches.

$$
\begin{array}{l}
\bullet\ \Diamond p, 1,\ \Diamond(p \wedge q), 1,\ \neg\neg q, 1 \\
| \\
\bullet\ 1r2, (p \wedge q),\ 2 \quad R_{\Diamond} \\
| \\
\bullet\ p, 2,\ q, 2 \quad R_{\wedge} \\
| \\
\bullet\ q, 1 \quad R_{\neg\neg}
\end{array}
$$

So we can see that we can move from the branch defined in our theory — the formalized branch — to the standard branch, and the other way round — from the intuitive branch to the formalized branch. However, moving in the other direction requires a more precise determination of what a branch is, as well as determination of what the tableau rules are. Nonetheless, these issues are clarified by the theory that has just been presented — so it is easier to apply the general concepts in practice.

Since we already have a certain method of applying the general concept of branch in practice, we can now describe the transition from formalized tableaux to standard ones.

We are still considering an arbitrary tableau system $\langle \mathsf{For}, \triangleright_{\mathbf{R}} \rangle$ — where For is a set of formulas, while **R** is a set of tableau rules — and set of tableau expressions Te on which the tableau rules from set **R** were defined.

Consider tableau $\langle X, A, \Phi \rangle$, where $X \subseteq \mathsf{For}$, $A \in \mathsf{For}$, whereas Φ is a set of branches — in accordance with general definition of tableau 5.48.

Based on set Φ we can define sets that only include translations of branches from set Φ — one for each branch from Φ. We impose condition: $(*)$ Ψ is such set of translations of branches from Φ that for each branch $\phi \in \Phi$, Ψ contains precisely one translation of ϕ. Since for each branch $\phi \in \Phi$, Ψ contains precisely one translation of ϕ, so the translation of branch ϕ in given set Ψ will be denoted as ϕ'.

Among the sets that meet condition $(*)$ there is at least one such set Φ' that the distribution of points and assigned in the translations included in set Φ' sets of expressions corresponds to the distribution of differences between the consequents and antecedents in the relevant branches contained in Φ as well as the inclusion of sub-branches of some branches in another ones.

We can carry out the following action on set Φ'. Define set $\langle \Phi'', \leq_{\Phi''} \rangle$:

1. $\Phi'' = \bigcup \Phi'$
2. $\langle n, X \rangle \leq_{\Phi''} \langle k, Y \rangle$ iff there exists such translation $\phi'_M \in \Phi'$ that $n, k \in M$, $\phi'_M(n) = X$, $\phi'_M(k) = Y$ and $n \leq_M k$.

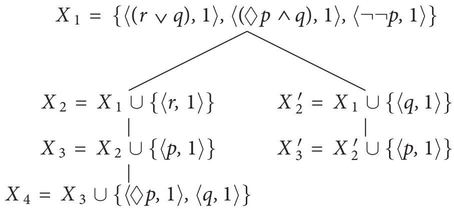
In set Φ'' there exists an element which is the smallest in terms of relation $\leq_{\Phi''}$ — that element is included in a set that is identical to the first element of each branch that belongs to set of branches Φ, i.e. the first step in the proof of fact that $X \triangleright A$.

Set $\langle \Phi'', \leq_{\Phi''} \rangle$ is an application of the definition of abstract concept of tableau $\langle X, A, \Phi \rangle$. In this way, to each tableau $\langle X, A, \Phi \rangle$ we can assign at least one tableau $\langle \Phi'', \leq_{\Phi''} \rangle$.

The transition from tableau $\langle X, A, \Phi \rangle$ to tableau $\langle \Phi'', \leq_{\Phi''} \rangle$ will be called *tableau translation* or simply *translation*.

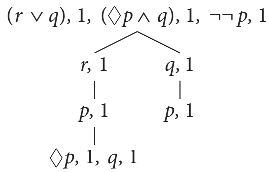
Here is an example. Again, consider set of tableau rules $\mathbf{R_{S5}}$ described in Chapter Four.

Example 6.50. Consider the following set of expressions $\{\langle (r \vee q), 1 \rangle, \langle \neg\neg p, 1 \rangle, \langle (\Diamond p \wedge q), 1 \rangle\}$. By the use of rule $R_{\vee}$, and then on left hand side — rules $R_{\neg\neg}$ and $R_{\wedge}$, and on the right have side — rules $R_{\neg\neg}$, we get the following branches.



In the light of definition of tableau 4.47, the set of these two branches is a tableau for pair $\langle \{(r \vee q), (\Diamond p \wedge q)\}, \neg p \rangle$, although it is not complete tableau.

And if we remove the brackets of sets and brackets from the tableau expressions, then we will get a graph of tableau corresponding to the graphs usually used to present tableaux.



This will be a result of some translation of set of branches Φ, i.e. some partially ordered set with the smallest element which is the root of the proof tree. We do not insert the denotations of points •. Instead of the denotations, there are individual lines with expressions.