Appendix: Optimising IS-LM model with endogenous investment and capital adjustment costs

A. Money demand (LM equation)

With respect to money demand, the period utility function (3.22) and equation (3.21) yield

\[ m_t = \exp\left[ \gamma^{-1} (\zeta_t - \nu_t) \right] c_t^\sigma \left[ \frac{Y_t}{(1 - \gamma)(1 + i_t)} \right]^{\frac{1}{\gamma}}. \]  

(A3.1)

After approximating \( i_t / (1 + i_t) \) by \( i_t \), (A3.1) takes the semi-log-linearised form

\[ \tilde{m}_t = \frac{\sigma}{\gamma} \tilde{c}_t - \frac{1}{\gamma} (i_t - \tilde{i}_t) + \frac{1}{\gamma} \left( \zeta_t - \nu_t \right). \]  

(A3.2)

In (A3.2), an expression for the composite disturbance \( \chi_t = (\zeta_t - \nu_t) / \gamma \) can be substituted to yield the LM equation of the system:

\[ \tilde{m}_t = \frac{\sigma}{\gamma} \tilde{c}_t - \frac{1}{\gamma} (i_t - \tilde{i}_t) + \chi_t, \]  

(A3.3)

where \( \chi_t = (\zeta_t - \nu_t) / \gamma \) is the composite disturbance and \( \zeta_t \) and \( \nu_t \) are real money balances and consumption preference shocks. In (A3.3) real money balances depend positively on a transaction variable (consumption) and negatively on an opportunity-cost variable.

B. Steady-state capital

The first-order condition with capital (3.30) reads

\[ 1 + r_t = \frac{1 - \delta - E_t C_{t+1} + E_t (rmc_{t+1} mpk_{t+1})}{1 + C_t}. \]

After substituting adjustment costs as in (3.29),

\[ 1 + \rho = \frac{1 - \delta + (1 - \delta) \Theta_t (\Theta_2 + 1) \delta^{\Theta_t} + \Theta_1 \Theta_2 \delta^{\Theta_t \cdot 1} + rmc \cdot mpk}{1 + \Theta_t (\Theta_2 + 1) \delta^{\Theta_t}} , \]  

(A3.4)

where \( \bar{r} = \rho \). After substitution, equation (A3.4) can be transformed into

\[ 1 + \rho = \frac{1 - \delta + (1 - \delta + \Theta_2) \bar{C} + rmc \cdot mpk}{1 + \bar{C}_1} , \]  

(A3.5)

where \( \bar{C} = \Theta_t \delta^{\Theta_t} \) is the unit adjustment cost in steady state and \( \bar{C}_1 = \Theta_1 (\Theta_2 + 1) \delta^{\Theta_t} \) is the marginal adjustment cost in steady state. From (A3.5), the following relation for the marginal product of capital in steady state can be derived:

\[ mpk = \rho (1 + \bar{C}_1) + \delta (1 + \bar{C}) . \]  

(A3.6)
Capital stock in steady state is then determined by substituting with a Cobb-
Douglas technology with steady-state labour of 1 in (A3.6):
\[ \alpha_{pp}\bar{k}_{stev} = \frac{\rho(1+C_1) + \delta(1+C_1)}{rmc}. \] (A3.7)

Equation (A3.7) yields the steady-state capital
\[ \bar{k} = \left[ \frac{\alpha_{pp}rmc}{\rho(1+C_1) + \delta(1+C_1)} \right]^{\frac{1}{1-stev}}. \] (A3.8)

C. Inflation and real wage equations under flexible prices and wages

When nominal rigidities are absent, the symmetry conditions \( P_t = P_t^d \) and \( W_t = W_t^d \) hold in equilibrium. Then, the price symmetry condition equations (3.11) and (3.14) imply a constant real marginal cost \( rmc_t \) given by:
\[ rmc_t = \frac{\omega_t}{A_t f_t(A_t^d,k_t)} = \frac{\theta - 1}{\theta}. \] (A3.9)

Similarly, the wage symmetry condition and (3.15) imply a constant ratio of the leisure-consumption marginal rate of substitution over the real wage
\[ \frac{U_{c_t}(c_t,m_t,l_t,v_t,\xi_t)}{\omega_t U_t(c_t,m_t,l_t,v_t,\xi_t)} = \frac{\theta_t - 1}{\theta_t}. \] (A3.10)

To recall, production technology is given by the Cobb-Douglas function
\[ f(A_t^d,k_t) = (A_t^d)^{1-stev} k_t^{stev}, \quad 0 < \alpha_{pp} < 1. \] (3.2)

Then, for \( W_t = W_t^d \), from (A3.10), the period utility function (3.22), the time constraint (3.7), the overall resources constraint (3.26)\( ^{133} \), and the production function (3.2) yield\( ^{134} \):
\[ \hat{\omega}_t = \left[ \frac{\tau n}{l(1-\alpha_{pp})} + \frac{\sigma}{\omega_c} \right] \hat{y}_t - \frac{\alpha \omega_c}{\omega_c} \hat{g}_t - \frac{\tau n}{l} A_t - v_t, \] (A3.11)

where \( \tau \) is a leisure risk aversion coefficient from the utility function.

D. Deriving the system’s reduced forms

The structural equations (3.25), (3.26), (3.32’) and (3.33)-(3.35), together with (3.36), (3.41), (3.39) or (3.38) and (3.41) can be reduced to a system of four equations, expressed in terms of inflation, consumption, capital and the real wage. Equation (3.35) can be written as

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133 For simplicity, \( \omega_{i} = 0 \) is assumed. Later in Chapter III, Subsection 4.1 the calibration is consistent with this assumption.
\[ \hat{\text{inv}}_t = \frac{1}{\delta} k_{t+1} - k_t. \] 

Equations (3.33) and (3.34) provide the algebraic terms to be substituted for \( rmc_c \) and \( mpk_c \) in (3.32') and (3.36):
\[
rmc_c = \omega_t - \frac{\alpha_{mp}}{1-\alpha_{mp}} \left( \hat{k}_t - \hat{\gamma}_t \right) \quad (3.33)
\]
\[
mpk_c = mpk(\hat{\gamma}_t - \hat{k}_t). \quad (3.34)
\]
Next, \( \hat{\gamma}_t \) can be substituted out according to equations (3.26) and (3.35')\(^{135}\):
\[
\hat{\gamma}_t = \hat{\gamma}_t \cdot C_t + \frac{\omega_{inv}}{\delta} k_{t+1} - \omega_{inv} \left( \frac{1-\delta}{\delta} \right) \hat{k}_t. \quad (3.26')
\]
From (3.41)
\[
r_t = i_t - E_t \pi_{t+1}
\]
and after rewriting (3.25) as
\[
r_t = \sigma \left( E_i \hat{c}_{t+1} - \hat{c}_i \right), \quad (3.25')
\]
the nominal interest rate can be substituted in the policy rule equation (3.38) or (3.39). After substituting \( \hat{\gamma}_t \) out according (3.26'), the interest-rate rule relation becomes\(^{136}\):
\[
E_t \hat{c}_{t+1} = \left( 1 + \frac{\hat{\lambda}^*_c}{\sigma} \right) \hat{c}_i + \frac{\hat{\lambda}^*_c}{\sigma} \frac{\omega_{inv}}{\delta} k_{t+1} - \frac{\hat{\lambda}^*_c}{\sigma} \frac{\omega_{inv}}{\delta} \hat{k}_i - \frac{1}{\sigma} E_t \pi_{t+1} + \frac{\hat{\lambda}^*_c}{\sigma} \pi_t. \quad (A3.12)
\]
Finally, equations (3.25'), (3.33), (3.34), (3.35'), (3.26') and (3.41) can be combined with (3.32'), (3.36), (3.37) and (A3.12) to yield the system’s reduced forms:
\[
k_{t+1} = \frac{1}{N} \left[ 2b_4 + \Xi b_1 (b_6 + \delta) - \Xi \omega_{inv} \frac{\lambda_*}{C} (b_1 b_7 - 1) - \Xi \omega_{inv} b_3 \left( b_1 b_7 + b_5 - 1 \right) \right] \hat{k}_{t+1} +
\]
\[
\frac{1}{N} \left[ -b_4 + \Xi b_1 \left( b_1 b_7 - 1 \right) (b_6 + \delta) + \Xi \lambda_* b_1 \left( b_1 b_7 - 1 \right) \right] \hat{k}_i - \frac{\Xi \omega_{inv} \delta}{N} \left[ b_1 \left( \lambda_* b_1 + 1 \right) + b_3 \left( b_1 b_7 + b_5 - 1 \right) - \lambda_* \right] \hat{c}_i +
\]
\[
- \frac{\Xi \delta}{N} \left( \lambda_*^\gamma - \frac{1}{\beta} \right) \left( b_1 b_7 - 1 \right) \pi_i - \frac{\Xi \delta}{N \beta} \left[ b_2 \left( b_2 b_7 + b_5 - 1 \right) + b_2 b_7 \right] \hat{\gamma}_i + \frac{\Xi \delta b_3}{N \beta} \omega_{inv} \quad (3.32'')
\]
\[
E_t \pi_{t+1} = \frac{1}{\beta} \hat{\pi}_t - \frac{\omega_{inv} b_9}{\delta} \hat{k}_{t+1} + b_3 \left( \frac{b_6}{\delta} + 1 \right) \hat{k}_i - \omega_{inv} b_9 \hat{c}_i - \frac{b_9}{\beta} \hat{\omega}, \quad (3.36')
\]
\[
E_t \hat{\omega}_{t+1} = \frac{b_6 + b_2}{\beta} \hat{\omega}_t - \frac{1}{\beta} \hat{\omega}_{t+1} + \omega_{inv} b_1 \frac{\lambda}{k_{t+1}} - \frac{b_1 (b_6 + \delta)}{\delta} \hat{k}_i + \omega_{inv} b_9 \hat{c}_i. \quad (3.37')
\]

\(^{135}\) Equation (3.26') does not include the term \( \omega_{inv} \), as in the calibration \( \omega_{inv} = 0 \).

\(^{136}\) Here \( \lambda_*^\gamma \) and \( \lambda_*^c \) denote the effective values assigned to inflation and the output gap in the monetary rule. For the Taylor rule (3.38) \( \lambda_*^\gamma = 1 + \lambda_*^\gamma \) and \( \lambda_*^c = \lambda_*^c \). For the policy specification (3.39) \( \lambda_*^\gamma = (1 - \lambda_4)(1 + \lambda_4) \) and \( \lambda_*^c = (1 - \lambda_4) \lambda_*^c \). For the purpose of the subsequent determinacy analysis no interest-rate smoothing is assumed, i.e. \( \lambda_* = 0 \). The possibility of pursuing interest-rate smoothing will be further considered when the implications of shocks are examined.
\[ E_{\hat{c}_{r+1}} = \left[ 1 + b_1 (\hat{x}_r + b_1) \right] \hat{c}_r + \frac{1}{\sigma} \left( \frac{1}{\sigma} - b_1 \right) (\hat{x}_r + b_3) \hat{k}_{r+1} - \frac{1}{\sigma \delta} \left[ b_1 (b_3 + \delta) + \hat{x}_r b_3 \right] \hat{k}_r + \frac{1}{\sigma} \left( \hat{x}_r \frac{1}{\beta} \right) \hat{\tau}_r + b_2 \frac{b_3}{\sigma \beta} \hat{\omega}_r, \]

(A3.12’)

where

\[ b_1 = mpk \left( \frac{rmc * \alpha_{PF}}{1 - \alpha_{PF}} + 1 \right) > 0 \]

\[ b_2 = \frac{(1 - \beta \eta_p) (1 - \eta_p) (1 - \alpha_{PF})}{\eta_p (1 - \alpha_{PF} + \alpha_{PF} \delta)}, \quad 0 < b_2 < 1 \]

\[ b_3 = \frac{\alpha_{PF} (1 - \beta \eta_p) (1 - \eta_p)}{\beta \eta_p (1 - \alpha_{PF} + \alpha_{PF} \delta)}, \quad 0 < b_3 < b_2 < 1 \quad \text{for} \quad 0 < \alpha_{PF} < \beta < 1 \]

\[ b_4 = \frac{1}{1 + \delta}, \quad 0 < b_4 < 1 \quad \text{for} \quad \delta > 0 \]

\[ b_5 = mpk * rmc > 0 \]

\[ b_6 = \omega_{mv} (1 - \delta), \quad 0 < b_6 < 1 \]

\[ b_7 = \frac{\omega}{\sigma} > 0 \]

\[ b_8 = \frac{\alpha_{PF}}{1 - \alpha_{PF}}, \quad 0 < b_8 < 1 \]

\[ b_9 = 1 + \beta > 1 \]

\[ N = \frac{1}{1 + \delta} + \Xi * mpk * \omega_{mv} \left( \frac{rmc * \alpha_{PF}}{1 - \alpha_{PF}} + 1 \right) > 0. \]

For convenience, the system (3.32’), (3.36’), (3.37’) and (A3.12’) can be re-written as:

\[ \hat{k}_{r+1} = a_1 \hat{k}_{r+1} + a_2 \hat{k}_r - a_3 \hat{c}_r - a_4 \hat{\tau}_r + a_6 \hat{\omega}_{r+1} \]

(A3.13)

\[ E_{t, \hat{c}_{r+1}} = -a_1 \hat{k}_{r+1} + a_6 \hat{c}_r + a_7 \hat{\tau}_r - a_{10} \hat{\omega}_r + a_{11} \hat{\omega}_{r+1} \]

(A3.14)

\[ E_{t, \hat{\omega}_{r+1}} = a_{12} \hat{k}_{r+1} - a_{13} \hat{k}_r + a_{14} \hat{c}_r + a_{15} \hat{\tau}_r - a_{16} \hat{\omega}_{r+1} \]

(A3.15)

\[ E_{t, \hat{c}_{r+1}} = a_{17} \hat{k}_{r+1} - a_{18} \hat{k}_r + a_{19} \hat{c}_r + a_{20} \hat{\tau}_r + a_{21} \hat{\omega}_r \]

(A3.16)

where

\[ a_1 = \frac{1}{N} \left[ 2 b_4 + \Xi b_3 (b_6 + \delta) - \Xi \omega_{mv} \lambda_{y} (b_6 b_7 - 1) \right] \]

\[ a_2 = \frac{1}{N} \left[ -b_4 + \Xi b_3 (b_6 b_7 + b_3 - 1) (b_6 + \delta) + \Xi \lambda_{y} b_3 (b_6 b_7 - 1) \right] \]

\[ a_3 = \frac{\Xi \delta}{N} \left( \frac{\lambda_{y} - 1}{\beta} \right) (b_6 b_7 - 1) \]

\[ a_4 = \frac{\Xi \omega_{mv} \delta}{N} \left[ b_1 (\lambda_{y} b_7 + b_3 (b_6 b_7 + b_3 - 1) - \lambda_{y} b_3 (b_6 b_7 - 1) \right] \]

\[ a_5 = \frac{\Xi \delta}{NB} \left[ b_2 (b_6 b_7 + b_3 - 1) + b_5 b_9 \right] \]
\[ a_6 = \frac{\Xi \delta b_3}{N \beta} \]
\[ a_7 = \frac{\omega_m b_3}{\delta} \]
\[ a_8 = b_3 \left( \frac{b_6}{\delta} + 1 \right) \]
\[ a_9 = \frac{1}{\beta} \]
\[ a_{10} = \omega_a b_3 \]
\[ a_{11} = \frac{b_2}{\beta} \]
\[ a_{12} = \frac{\omega_m b_3}{\delta} \]
\[ a_{13} = \frac{b_1 (b_6 + \delta)}{\delta} \]
\[ a_{14} = \omega_e b_3 \]
\[ a_{15} = \frac{b_5 + b_6}{\beta} \]
\[ a_{16} = \frac{1}{\beta} \]
\[ a_{17} = \frac{1}{\delta} \left( \frac{1}{\sigma} - b_7 \right) \left( \lambda^*_y + b_3 \right) \]
\[ a_{18} = \frac{1}{\sigma \delta} \left[ b_3 (b_6 + \delta) + \lambda^*_y b_6 \right] \]
\[ a_{19} = \frac{1}{\sigma} \left( \lambda^*_x - \frac{1}{\beta} \right) \]
\[ a_{20} = \left[ 1 + b_7 \left( \lambda^*_y + b_3 \right) \right] \]
\[ a_{21} = \frac{b_2}{\sigma \beta}. \]